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PERTURBATION SOLUTIONS OF THE ELLIPSOIDAL WAVE EQUATION

By F. M. ARSCOTT (*Makerere, Uganda*)

[Received 1 November 1955]

1. Introduction

WHEN we separate the wave equation in ellipsoidal coordinates, we are led to the following ordinary linear differential equation:

$$\frac{d^2w}{dz^2} = (a+bk^2\mathrm{sn}^2z+qk^4\mathrm{sn}^4z)w, \tag{1}$$

where the Jacobian elliptic functions are formed with modulus k; q is a determinate constant, depending on the wave number, and a and b are arbitrary constants which we have to dispose of in such a way that the equation (1) shall possess a doubly-periodic uniform solution with real period $2K$ or $4K$, and imaginary period $2iK'$ or $4iK'$. It is to such solutions only that I shall give the name of 'ellipsoidal wave functions'; solutions with other periods exist but do not normally satisfy the necessary conditions for being physically significant.

The equation (1) has been known in this context certainly since Möglich's paper of 1927 (**1**) but has proved considerably less tractable than other equations with periodic or doubly-periodic coefficients, and the only other substantial contribution to the theory of such functions has been made by Malurkar (**2**). Apart from a short section in Strutt's 'Lamésche, Mathieusche und verwandte Funktionen' (**3**) and one paper by Campbell (**4**), no further work appears to have been done on the equation. It is, however, of considerable interest from the purely mathematical point of view, apart from its physical significance, since it includes, as special or limiting cases, such well-known equations as those of Lamé and Mathieu, the associated Mathieu equation (or spheroidal wave equation), and Hill's equation with three terms.

The perturbation method of solution, historically the earliest used for the solution of differential equations with periodic coefficients, goes back to Mathieu's work in 1868 on the equation which bears his name. It consists in assuming a solution of the equation, possessing the necessary periodic properties, in the form of a series of powers of one of the

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parameters of the equation (in this case q), the coefficients of this series being functions of the independent variable; at the same time the other parameters of the equation (in this case a and b) are also assumed to be expressed as power series in the first parameter q. Having made these assumptions, the coefficients in the various series are found successively by rejecting all terms not possessing the necessary periodic properties.

The essential disadvantage of the perturbation method is that it is very difficult to determine the convergence of the resulting series. Even for the comparatively simple perturbation solutions of Mathieu's equation the problem has not been fully solved, and it is not yet possible to state any lower limit to the radius of convergence of the series which will be obtained in this paper. For this reason, the use of perturbation solutions for the other differential equations with periodic coefficients has now been largely abandoned, particularly with the development of the technique for dealing with series whose coefficients are given by a three-term recursion system. In the case of the equation (1), however, no such series has yet been discovered, and, although the author in some unpublished work has developed a somewhat similar series, it makes extensive use of Lamé polynomials which so far are very inadequately tabulated.

In spite, therefore, of the disadvantages of the perturbation solutions, they are, in fact, virtually the only explicit solutions of equation (1) so far known. In this paper, I give not only the results so far obtained, but sufficient of the intermediate working to make further development of the solutions as easy as possible.

In the heavy calculations which this method involves, the principal feature is the indefinite integral $\int \mathrm{sn}^{2n}z\,dz$ for positive integral values of n. From the formula

$$\frac{d}{dz}\{\mathrm{sn}^{n-1}z\,\mathrm{cn}\,z\,\mathrm{dn}\,z\} = (n-1)\mathrm{sn}^{n-2}z-n(1+k^2)\mathrm{sn}^n z+(n+1)k^2\mathrm{sn}^{n+2}z,$$

and writing $\int \mathrm{sn}^{2n}z\,dz = I_{2n}$, we obtain the recurrence formula

$$(n-1)I_{n-2}-n(1+k^2)I_n+(n+1)k^2I_{n+2} = \mathrm{sn}^{n-1}z\,\mathrm{cn}\,z\,\mathrm{dn}\,z, \qquad (2)$$

by which we can express I_4, I_6, etc., in terms of I_0 and I_2. In fact,

$$I_0 = z, \qquad I_2 = \frac{z-E(z)}{k^2},$$

but, since these terms are not doubly-periodic, their precise values are not needed.

Using (2) we obtain the following expressions, where for brevity we write s, c, d for $\operatorname{sn} z$, $\operatorname{cn} z$, $\operatorname{dn} z$.

$$3k^2I_4 = scd+2(1+k^2)I_2-I_0, \tag{3a}$$

$$5k^2I_6 = s^3cd+\frac{4(1+k^2)}{3k^2}scd+\frac{8+7k^2+8k^4}{3k^2}I_2-\frac{4(1+k^2)}{3k^2}I_0, \tag{3b}$$

$$\begin{aligned}7k^2I_8 = s^5cd&+\frac{6(1+k^2)}{5k^2}s^3cd+\frac{24+23k^2+24k^4}{15k^4}scd+\\&+\frac{8(1+k^2)(6-k^2+6k^4)}{15k^4}I_2-\frac{24+23k^2+24k^4}{15k^4}I_0,\end{aligned} \tag{3c}$$

$$\begin{aligned}9k^2I_{10} = s^7cd&+\frac{8(1+k^2)}{7k^2}s^5cd+\frac{48+47k^2+48k^4}{35k^4}s^3cd+\\&+\frac{4(1+k^2)(16-k^2+16k^4)}{35k^6}scd+\\&+\frac{128+104k^2+99k^4+104k^6+128k^8}{35k^6}I_2-\\&-\frac{4(1+k^2)(16-k^2+16k^4)}{35k^6}I_0,\end{aligned} \tag{3d}$$

$$\begin{aligned}11k^2I_{12} = s^9cd&+\frac{10(1+k^2)}{9k^2}s^7cd+\frac{80+79k^2+80k^4}{63k^4}s^5cd+\\&+\frac{16(1+k^2)(30-k^2+30k^4)}{315k^6}s^3cd+\\&+\frac{640+592k^2+579k^4+592k^6+640k^8}{315k^8}scd+\\&+\frac{(1+k^2)(1280-256k^2+1206k^4-256k^6+1280k^8)}{315k^8}I_2-\\&-\frac{640+592k^2+579k^4+592k^6+640k^8}{315k^8}I_0.\end{aligned} \tag{3e}$$

We notice (it is useful later) that the coefficients of scd and I_0 are in every case the same but with opposite signs.

2. Notation

When $q = 0$, the equation (1) reduces to

$$\frac{d^2w}{dz^2} = (a+bk^2\operatorname{sn}^2 z)w,$$

which is Lamé's equation; it is known that, for this equation to possess doubly-periodic uniform solutions with periods $2K$ or $4K$, $2iK'$ or $4iK'$, we must have $b = n(n+1)$, where n is an integer which may be taken positive, and a must have a characteristic value, of which there are $2n+1$ for a given value of n. Then the required solutions are the well-known Lamé polynomials; the solutions of (1) which we require must therefore be such as to reduce to Lamé polynomials when $q = 0$.

It is possible to prove that (i) to every Lamé polynomial there corresponds one and only one solution of equation (1) which is doubly-periodic (with periods $2K$ or $4K$, $2iK'$ or $4iK'$) and uniform, and which is such as to reduce to that Lamé polynomial when $q = 0$, (ii) there are no other solutions of (1) which are doubly-periodic (with these periods) and uniform. In other words, the correspondence between Lamé polynomials and ellipsoidal wave functions (as we have defined them) is one-to-one.

(The proofs of these statements have been obtained by the author in some so far unpublished work, but are lengthy since they depend on the construction of ellipsoidal wave functions by a quite different method from that used in this paper.)

This makes it possible to base our classification of, and notation for, ellipsoidal wave functions on the classification and notation used for Lamé polynomials; it is convenient, however, to make first a slight extension to the standard notation for the latter. In proposing this, the notation used by E. L. Ince (**5**) has not been overlooked; the latter, however, was introduced primarily to deal with solutions of Lamé's equation when n is not integral, and, while applicable to the Lamé polynomials, is somewhat cumbersome when so applied. The predominance of the polynomials, both in practical applications and in the theory of ellipsoidal wave functions, is so great as to justify, in the author's view, a separate notation. In any case the correspondence between Ince's notation and that used here is quite simple and is shown in Table II below.

By Lamé polynomials are meant those solutions of

$$\frac{d^2w}{dz^2} = \{-h+n(n+1)k^2\text{sn}^2z\}w \tag{4}$$

which are of the form $\text{sn}^r z\, \text{cn}^s z\, \text{dn}^t z\, F(\text{sn}^2 z)$ where r, s, t may be 0 or 1 and $F(\text{sn}^2z)$ is a polynomial in sn^2z. There are thus eight different types, distinguished by the values of r, s, t and consequently by properties of periodicity and parity. I shall retain the general symbol $E_n^m(z)$ to denote any Lamé polynomial of degree n, but distinguish the eight types by

prefixing, when necessary, the letters u, s, c, d, sc, sd, cd, scd according as the function $F(\mathrm{sn}^2 z)$ is multiplied by unity, $\mathrm{sn}\,z$, $\mathrm{cn}\,z$, $\mathrm{dn}\,z$, $\mathrm{sn}\,z\,\mathrm{cn}\,z$, $\mathrm{sn}\,z\,\mathrm{dn}\,z$, $\mathrm{cn}\,z\,\mathrm{dn}\,z$, $\mathrm{sn}\,z\,\mathrm{cn}\,z\,\mathrm{dn}\,z$, respectively. For example, an expression of the above form with $r = 0$, $s = t = 1$, would be written as $cdE_n^m(z)$. I specify the upper index m as the number of zeros in the range

$$0 < z < K.$$

It is often more convenient, however, to write the lower suffix, not as n but as $2n$, $2n+1$, $2n+2$, $2n+3$ according as the polynomial is of the first, second, third, or fourth species. Then the eight types, with their elementary properties, are denoted thus:

TABLE I

Type	*Parity*	*Periods*	*Type*	*Parity*	*Periods*
uE_{2n}^m	Even	$2K,\ 2iK'$	scE_{2n+2}^m	Odd	$2K,\ 4iK'$
sE_{2n+1}^m	Odd	$4K,\ 2iK'$	sdE_{2n+2}^m	Odd	$4K,\ 4iK'$
cE_{2n+1}^m	Even	$4K,\ 4iK'$	cdE_{2n+2}^m	Even	$4K,\ 2iK'$
dE_{2n+1}^m	Even	$2K,\ 4iK'$	$scdE_{2n+3}^m$	Odd	$2K,\ 2iK'$

The advantages of this numeration are that

(i) the polynomial $F(\mathrm{sn}^2 z)$ is in every case of degree n in $\mathrm{sn}^2 z$;

(ii) the ranges of the n, m are in every case $n = 0, 1, 2,..., m = 0, 1, 2,..., n$;

(iii) each of the above functions possesses m zeros in $0 < z < K$ and $n-m$ zeros in $K < z < K+iK'$. The former statement is the definition of m, the latter follows from Erdélyi's results in (**6**).

The correspondence between this notation and that of Ince is easily seen to be as follows. The upper index in Ince's notation denotes the number of zeros in the range $-K < z \leqslant K$; constant factors are ignored.

TABLE II

$$Ec_{2n}^{2m} = uE_{2n}^m; \quad Ec_{2n+1}^{2m} = dE_{2n+1}^m; \quad Es_{2n}^{2m+2} = scE_{2n+2}^m; \quad Es_{2n+1}^{2m+2} = scdE_{2n+}^m$$
$$Ec_{2n}^{2m+1} = cdE_{2n+2}^m; \quad Ec_{2n+1}^{2m+1} = cE_{2n+1}^m; \quad Es_{2n}^{2m+1} = sdE_{2n+2}^m; \quad Es_{2n+1}^{2m+1} = sE_{2n+1}^m.$$

With this notation for Lamé polynomials, we simply classify ellipsoidal wave functions according to the Lamé polynomials to which they reduce when $q = 0$. I use the general symbol $\mathrm{el}(z)$ to denote *any* ellipsoidal wave function and $\mathrm{el}_n^m(z)$ to denote that ellipsoidal wave function which reduces to $E_n^m(z)$ when $q = 0$. I then further prefix the letters u, s, c, d, etc., to the symbol $\mathrm{el}_n^m(z)$, when needed, in exactly the same way, so that, for example, $sd\,\mathrm{el}_6^2(z)$ is that function which reduces to $sdE_6^2(z)$ for $q = 0$. We thus have eight types of ellipsoidal wave function, whose properties of periodicity and parity are the same as the Lamé polynomials of the corresponding types.

The values of a and b corresponding to a given ellipsoidal wave function can be denoted (though it will often not be necessary) by using the same n and m suffixes and putting the letters denoting the type in brackets, thus for the function $sd\,\mathrm{el}_6^2(z)$ we denote them by $a^{(sd)2}_{6}$ and $b^{(sd)2}_{6}$.

The functions thus defined are determinate except for a constant factor. For the calculation of perturbation solutions (though not for the development of the general theory) the most convenient method of normalization is as follows. If $E_n^m(z)$ and $\mathrm{el}_n^m(z)$ are written in the forms

$$\mathrm{sn}^r z\,\mathrm{cn}^s z\,\mathrm{dn}^t z \sum_{p=0}^{N} A_p\,\mathrm{sn}^{2p} z$$

and

$$\mathrm{sn}^r z\,\mathrm{cn}^s z\,\mathrm{dn}^t z \sum_{p=0}^{\infty} B_p\,\mathrm{sn}^{2p} z$$

respectively, then we normalize $E_n^m(z)$ by the stipulation that $A_N = 1$, and $\mathrm{el}_n^m(z)$ by the stipulation that $B_N = 1$ for *all* values of q.

The calculation of the first ellipsoidal wave-function of each type, that is to say those which for $q = 0$ reduce to $1, \mathrm{sn}\,z, \mathrm{cn}\,z,\ldots, \mathrm{sn}\,z\,\mathrm{cn}\,z\,\mathrm{dn}\,z$, respectively, is simpler than the more general case because (as will be seen) it involves only integration while the more general case involves finding particular integrals of certain linear differential equations. We therefore deal first with this special case.

3. Perturbation solutions for the first ellipsoidal wave function of each type

Consider first the construction of $u\,\mathrm{el}_0^0(z)$. For $q = 0$, this reduces to $E_0^0(z) \equiv 1$, and a and b both reduce to 0. We therefore set

$$w = 1 + \sum q^t f_t(z), \qquad a = \sum c_t q^t, \qquad b = \sum d_t q^t,$$

and substitute in (1). Comparing the coefficients of $q, q^2, q^3,\ldots$, we obtain the sequence of equations

$$f_1''(z) = c_1 + d_1 k^2\,\mathrm{sn}^2 z + k^4\,\mathrm{sn}^4 z,$$

$$\begin{aligned} f_2''(z) &= (c_1 + d_1 k^2\,\mathrm{sn}^2 z + k^4\,\mathrm{sn}^4 z) f_1(z) + c_2 + d_2 k^2\,\mathrm{sn}^2 z \\ &= f_1(z) f_1''(z) + c_2 + d_2 k^2\,\mathrm{sn}^2 z, \end{aligned}$$

$$\begin{aligned} f_3''(z) &= (c_1 + d_1 k^2\,\mathrm{sn}^2 z + k^4\,\mathrm{sn}^4 z) f_2(z) + (c_2 + d_2 k^2\,\mathrm{sn}^2 z) f_1(z) + c_3 + d_3 k^2\,\mathrm{sn}^2 z \\ &= f_2(z) f_1''(z) + f_1(z)\{f_2''(z) - f_1(z) f_1''(z)\} + c_3 + d_3 k^2\,\mathrm{sn}^2 z, \text{ etc.} \end{aligned}$$

We solve these successively using the formulae (3), and choosing the c's and d's in such a way as to exclude from $f_t(z)$ all terms which are not doubly periodic. From the first equation we have, using (3 a),

$$f_1'(z) = (c_1 - \tfrac{1}{3}k^2) I_0 + k^2\{d_1 + \tfrac{2}{3}(1+k^2)\} I_2 + \tfrac{1}{3}k^2\,\mathrm{sn}\,z\,\mathrm{cn}\,z\,\mathrm{dn}\,z + \text{constant};$$

but, if $f_1(z)$ is doubly-periodic, $f_1'(z)$ must be doubly-periodic also and the coefficients of I_0 and I_2 (which are not doubly-periodic) must vanish. The constant of integration must also be zero or $f_1(z)$ would not be doubly-periodic but would contain a term in z. Hence we have

$$c_1 = \tfrac{1}{3}k^2, \qquad d_1 = -\tfrac{2}{3}(1+k^2), \qquad f_1'(z) = \tfrac{1}{3}k^2 \operatorname{sn} z \operatorname{cn} z \operatorname{dn} z.$$

Integration gives $\qquad f_1(z) = \tfrac{1}{6}k^2 \operatorname{sn}^2 z + \text{constant},$

but the constant of integration must again be taken as zero or the normalization condition will be broken (in fact, none of the $f_i(z)$ will contain a constant term).

The values of $f_1(z)$, c_1, d_1 thus obtained are then substituted in the next equation and $f_2(z)$, c_2, d_2 similarly calculated. There are no noteworthy features of the calculation, and the results are given in full in Table III.

To apply the same process to the construction of the first function of each of the other types, it is convenient to make first a preliminary transformation of the equation to remove the extraneous factors $\operatorname{sn} z$, $\operatorname{cn} z$, $\operatorname{dn} z$. These transformations are as follows; w represents a solution of the standard equation

$$\frac{d^2w}{dz^2} = (a+bk^2 \operatorname{sn}^2 z+qk^4 \operatorname{sn}^4 z)w.$$

If $w = y \operatorname{sn} z$, then

$$\frac{d}{dz}\left(\operatorname{sn}^2 z \frac{dy}{dz}\right) = [(1+k^2+a)\operatorname{sn}^2 z+(b-2)k^2 \operatorname{sn}^4 z+qk^4 \operatorname{sn}^6 z]y. \tag{5 a}$$

If $w = y \operatorname{cn} z$, then

$$\frac{d}{dz}\left(\operatorname{cn}^2 z \frac{dy}{dz}\right)$$
$$= [1+a+\{(b-2)k^2-1-a\}\operatorname{sn}^2 z+(2-b+qk^2)k^2 \operatorname{sn}^4 z-qk^4 \operatorname{sn}^6 z]y. \tag{5 b}$$

If $w = y \operatorname{dn} z$, then

$$\frac{d}{dz}\left(\operatorname{dn}^2 z \frac{dy}{dz}\right)$$
$$= [a+k^2+(b-2-a-k^2)k^2 \operatorname{sn}^2 z+(q-b+2)k^4 \operatorname{sn}^4 z-qk^6 \operatorname{sn}^6 z]y. \tag{5 c}$$

If $w = y \operatorname{sn} z \operatorname{cn} z$, then

$$\frac{d}{dz}\left(\operatorname{sn}^2 z \operatorname{cn}^2 z \frac{dy}{dz}\right)$$
$$= [(4+k^2+a)\operatorname{sn}^2 z+\{(b-6)k^2-4-k^2-a\}\operatorname{sn}^4 z+$$
$$+(qk^2-b+6)k^2 \operatorname{sn}^6 z-qk^4 \operatorname{sn}^8 z]y. \tag{5 d}$$

If $w = y\,\mathrm{sn}\,z\,\mathrm{dn}\,z$, then

$$\frac{d}{dz}\left(\mathrm{sn}^2z\,\mathrm{dn}^2z\,\frac{dy}{dz}\right)$$
$$= [(1+4k^2+a)\mathrm{sn}^2z+(b-7-4k^2-a)k^2\,\mathrm{sn}^4z+$$
$$+(q-b+6)k^4\,\mathrm{sn}^6z-qk^6\mathrm{sn}^8z]y. \quad (5\,\mathrm{e})$$

If $w = y\,\mathrm{cn}\,z\,\mathrm{dn}\,z$, then

$$\frac{d}{dz}\left(\mathrm{cn}^2z\,\mathrm{dn}^2z\,\frac{dy}{dz}\right)$$
$$= [(1+k^2+a)+\{(b-6)k^2-(1+k^2)(1+k^2+a)\}\mathrm{sn}^2z+$$
$$+\{qk^2-(b-6)(1+k^2)+1+k^2+a\}k^2\,\mathrm{sn}^4z+$$
$$+\{b-6-q(1+k^2)\}k^4\,\mathrm{sn}^6z+qk^6\,\mathrm{sn}^8z]y. \quad (5\,\mathrm{f})$$

If $w = y\,\mathrm{sn}\,z\,\mathrm{cn}\,z\,\mathrm{dn}\,z$, then

$$\frac{d}{dz}\left(\mathrm{sn}^2z\,\mathrm{cn}^2z\,\mathrm{dn}^2z\,\frac{dy}{dz}\right)$$
$$= [(4+4k^2+a)\mathrm{sn}^2z+\{(b-12)k^2-(4+4k^2+a)(1+k^2)\}\mathrm{sn}^4z+$$
$$+\{qk^2-(b-12)(1+k^2)+4+4k^2+a\}k^2\,\mathrm{sn}^6z+$$
$$+\{b-12-q(1+k^2)\}k^4\,\mathrm{sn}^8z+qk^6\,\mathrm{sn}^{10}z]y. \quad (5\,\mathrm{g})$$

The calculation follows in every case the same general lines as for the function $u\,\mathrm{el}_0^0(z)$, but there are certain devices which are not obvious and which shorten the working considerably; I shall therefore illustrate them by working through the expansion of $d\,\mathrm{el}_1^0(z)$ in some detail.

Since $dE_1^0(z) \equiv \mathrm{dn}\,z$, and since, for $\mathrm{dn}\,z$ to be a solution of (1) with $q = 0$, we must have $a = -k^2$ and $b = 2$, we assume the solution of (5 c) in the form

$$y = 1+\sum q^t f_t, \qquad a = -k^2+\sum c_t q^t, \qquad b = 2+\sum d_t q^t,$$

substitute in the equation and compare coefficients of powers of q. From the coefficients of q we have

$$\frac{d}{dz}(\mathrm{dn}^2z\,f_1') = c_1+(d_1-c_1)k^2\,\mathrm{sn}^2z+(1-d_1)k^4\,\mathrm{sn}^4z-k^6\,\mathrm{sn}^6z,$$

i.e.
$$\mathrm{dn}^2z\,f_1' = c_1 I_0+(d_1-c_1)k^2I_2+(1-d_1)k^4I_4-k^6I_6. \quad (6)$$

Instead of finding c_1 and d_1 first, we assume $f_1 = A\,\mathrm{sn}^2z$, where A is a constant to be determined, so that

$$\mathrm{dn}^2z\,f_1' = 2Ascd-2k^2As^3cd.$$

Now we expand the right-hand side of (6), using formulae (3 a), (3 b), and compare coefficients of s^3cd, scd, I_2, and I_0, getting the four equations

$$\begin{aligned} -2Ak^2 &= -\tfrac{1}{5}k^4, \\ 2A &= \tfrac{1}{3}(1-d_1)k^2-\tfrac{4}{15}k^2(1+k^2), \\ 0 &= (d_1-c_1)k^2+\tfrac{2}{3}k^2(1+k^2)(1-d_1)-\tfrac{1}{15}k^2(8+7k^2+8k^4), \\ 0 &= c_1-\tfrac{1}{3}(1-d_1)k^2+\tfrac{4}{15}k^2(1+k^2). \end{aligned}$$

The first of these gives $A = \frac{1}{10}k^2$; the second then gives $d_1 = -\frac{2}{5}(1+2k^2)$; c_1 could be obtained from either of the last two equations, but much work is saved by combining the second and fourth equations, giving $c_1 = 2A = \frac{1}{5}k^2$. The third equation may then be used for checking. The possibility of eliminating d_1 between the second and fourth equations results from the fact noted that the coefficients of I_0 and scd in the formulae (3) are numerically equal.

(This procedure is possible only when expanding functions which do not contain $\operatorname{sn} z$ as a factor. It appears to be true in the other cases that $c_1 = 6A$, but this is not immediately apparent.)

Having thus found the values of c_1 and d_1, and that $f_1 = \frac{1}{10}k^2 \operatorname{sn}^2 z$, we use the next equation, which is

$$\frac{d}{dz}(\operatorname{dn}^2 z f_2') = c_2+(d_2-c_2)k^2 \operatorname{sn}^2 z-d_2 k^4 \operatorname{sn}^4 z+f_1\frac{d}{dz}(\operatorname{dn}^2 z f_1').$$

To solve this, we assume

$$f_2 = B\operatorname{sn}^4 z+C\operatorname{sn}^2 z,$$

and on substituting this and the known expression for f_1 we compare coefficients of s^5cd and s^3cd, giving

$$-4k^2B = -\frac{k^6}{70}, \qquad 2(2B-k^2C) = k^4\left(\frac{-6(1+k^2)}{350}+\frac{7+4k^2}{250}\right).$$

These easily yield

$$B = \frac{k^4}{280}, \qquad C = \frac{k^2(3+k^2)}{1750}.$$

From the two equations arising from the coefficients of scd and I_0 we at once find

$$c_2 = 2C = \frac{k^2(3+k^2)}{875}.$$

Then either of these two equations may be used to find

$$d_2 = \frac{-2(3-3k^2+2k^4)}{875}.$$

The remaining equation, that obtained from the coefficients of I_2, may be used for checking.

Proceeding on the above lines, the following eight solutions have been developed as far as the term which is $O(q^2)$, and in the case of $u\,\mathrm{el}_0^0$ as far as $O(q^3)$. There seems to be no difficulty apart from the complicated working in extending them further. The work becomes more complicated with the greater number of extraneous factors involved; for example, in calculating the term which is $O(q^2)$ in $u\,\mathrm{el}_0^0$ we need only the formulae (3a) and (3b): that is to say, the formulae for I_4 and I_6; to calculate the corresponding term in $scd\,\mathrm{el}_3^0$ we need the formulae up to and including I_{12}.

Table III

[The terms omitted are $O(q^3)$ in every case except (1) when they are $O(q^4)$]

(1) $$u\,\mathrm{el}_0^0(z) = 1+\frac{qk^2\,\mathrm{sn}^2 z}{6}+q^2\left[\frac{k^4\,\mathrm{sn}^4 z}{120}+\frac{k^2(1+k^2)\mathrm{sn}^2 z}{270}\right]+$$
$$+q^3\left[\frac{k^6\,\mathrm{sn}^6 z}{2^4.3^2.5.7}+\frac{k^4(1+k^2)\mathrm{sn}^4 z}{2^2.3^3.5.7}-\frac{k^2(1-4k^2+k^4)\mathrm{sn}^2 z}{3^5.5.7}\right].$$

$$a^{(u)}{}_0^0 = q\frac{k^2}{3}+q^2\frac{k^2(1+k^2)}{135}-q^3\frac{2k^2(1-4k^2+k^4)}{3^5.5.7},$$

$$b^{(u)}{}_0^0 = -q\frac{2(1+k^2)}{3}-q^2\frac{2(1-k^2+k^4)}{135}-q^3\frac{2(1+k^2)(2-5k^2+2k^4)}{3^5.5.7}.$$

(2) $$s\,\mathrm{el}_1^0(z) = \mathrm{sn}\,z+q\frac{k^2\,\mathrm{sn}^3 z}{10}+q^2\,\mathrm{sn}\,z\left[\frac{k^4\,\mathrm{sn}^4 z}{280}+\frac{k^2(1+k^2)\mathrm{sn}^2 z}{1750}\right],$$

$$a^{(s)}{}_1^0 = -1-k^2+q\frac{3k^2}{5}+q^2\frac{3k^2(1+k^2)}{875},$$

$$b^{(s)}{}_1^0 = 2-q\frac{4(1+k^2)}{5}-q^2\frac{2(2-k^2+2k^4)}{875},$$

(3) $$c\,\mathrm{el}_1^0(z) = \mathrm{cn}\,z+q\,\mathrm{cn}\,z\frac{k^2\,\mathrm{sn}^2 z}{10}+q^2\,\mathrm{cn}\,z\left[\frac{k^4\,\mathrm{sn}^4 z}{280}+\frac{k^2(1+3k^2)\mathrm{sn}^2 z}{1750}\right],$$

$$a^{(c)}{}_1^0 = -1+q\frac{k^2}{5}+q^2\frac{k^2(1+3k^2)}{875},$$

$$b^{(c)}{}_1^0 = 2-q\frac{2(2+k^2)}{5}-q^2\frac{2(2-3k^2+3k^4)}{875},$$

(4) $$d\,\mathrm{el}_1^0(z) = \mathrm{dn}\,z+q\,\mathrm{dn}\,z\frac{k^2\,\mathrm{sn}^2 z}{10}+q^2\,\mathrm{dn}\,z\left[\frac{k^4\,\mathrm{sn}^4 z}{280}+\frac{k^2(3+k^2)\mathrm{sn}^2 z}{1750}\right].$$

$$a^{(d)}{}_1^0 = -k^2+q\frac{k^2}{5}+q^2\frac{k^2(3+k^2)}{875},$$

$$b^{(d)}{}_1^0 = 2-q\frac{2(1+2k^2)}{5}-q^2\frac{2(3-3k^2+2k^4)}{875},$$

TABLE III (*cont.*)

(5) $$sc\,\mathrm{el}_2^0(z) = \operatorname{sn}z\operatorname{cn}z + q\operatorname{sn}z\operatorname{cn}z\frac{k^2\operatorname{sn}^2 z}{14} + q^2\operatorname{sn}z\operatorname{cn}z\left[\frac{k^4\operatorname{sn}^4 z}{504} + \frac{k^2(1+3k^2)\operatorname{sn}^2 z}{2.3^2.7^3}\right],$$

$$a^{(sc)0}_{\ 2} = -4-k^2+q\frac{3k^2}{7}+q^2\frac{k^2(1+3k^2)}{3.7^3},$$

$$b^{(sc)0}_{\ 2} = 6-q\frac{2(2k^2+3)}{7}-q^2\frac{2(1-k^2+2k^4)}{3.7^3},$$

(6) $$sd\,\mathrm{el}_2^0(z) = \operatorname{sn}z\operatorname{dn}z + q\operatorname{sn}z\operatorname{dn}z\frac{k^2\operatorname{sn}^2 z}{14} + q^2\operatorname{sn}z\operatorname{dn}z\left[\frac{k^4\operatorname{sn}^4 z}{504} + \frac{k^2(3+k^2)\operatorname{sn}^2 z}{2.3^2.7^3}\right],$$

$$a^{(sd)0}_{\ 2} = -1-4k^2+q\frac{3k^2}{7}+q^2\frac{k^2(3+k^2)}{3.7^3},$$

$$b^{(sd)0}_{\ 2} = 6-q\frac{2(2+3k^2)}{7}-q^2\frac{2(2-k^2+k^4)}{3.7^3},$$

(7) $$cd\,\mathrm{el}_2^0(z) = \operatorname{cn}z\operatorname{dn}z + q\operatorname{cn}z\operatorname{dn}z\frac{k^2\operatorname{sn}^2 z}{14} + q^2\operatorname{cn}z\operatorname{dn}z\left[\frac{k^4\operatorname{sn}^4 z}{504} + \frac{k^2(1+k^2)\operatorname{sn}^2 z}{2.3.7^3}\right],$$

$$a^{(cd)0}_{\ 2} = -1-k^2+q\frac{k^2}{7}+q^2\frac{k^2(1+k^2)}{3.7^3},$$

$$b^{(cd)0} = 6-q\frac{4(1+k^2)}{7}-q^2\frac{2(2-3k^2+2k^4)}{3.7^3},$$

(8) $$scd\,\mathrm{el}_3^0(z) = \operatorname{sn}z\operatorname{cn}z\operatorname{dn}z + q\operatorname{sn}z\operatorname{cn}z\operatorname{dn}z\frac{k^2\operatorname{sn}^2 z}{18} +$$

$$+q^2\operatorname{sn}z\operatorname{cn}z\operatorname{dn}z\left[\frac{k^4\operatorname{sn}^4 z}{2^3.3^2.11} + \frac{k^2(1+k^2)\operatorname{sn}^2 z}{2.3^5.11}\right],$$

$$a^{(scd)0}_{\ 3} = -4-4k^2+q\frac{k^2}{3}+q^2\frac{k^2(1+k^2)}{3^4.11},$$

$$b^{(scd)0}_{\ 3} = 12-q\frac{2(1+k^2)}{3}-q^2\frac{2(1-k^2+k^4)}{3^4.11}.$$

4. Perturbation solutions for other functions

As has been mentioned above, the perturbation solutions for ellipsoidal wave functions other than those in Table III are more complex. The complexity arises, however, less from the solution itself than from the fact that the Lamé polynomials themselves are complicated expressions; this makes the working very difficult in the general case, where the value of k^2 is arbitrary. If, however (as is normally the case in practical applications) the numerical value of k^2 is known *a priori*, much of the difficulty disappears.

I shall illustrate the calculation by considering the function $u\,\mathrm{el}_2^0(z)$. The Lamé polynomial to which this reduces for $q = 0$, namely $uE_2^0(z)$ is

$$uE_2^0(z) = \operatorname{sn}^2 z - \frac{1+k^2+\sqrt{(1-k^2+k^4)}}{3k^2}, \tag{7}$$

which satisfies the equation

$$\frac{d^2w}{dz^2} = [-2(1+k^2)+2\surd(1-k^2+k^4)+6k^2\,\mathrm{sn}^2z]w. \tag{8}$$

For brevity, write

$$\lambda \equiv \surd(1-k^2+k^4), \qquad Q \equiv \frac{1+k^2+\lambda}{3k^2}, \qquad a_0 = -2(1+k^2)+2\lambda,$$

so that $uE_2^0(z) = \mathrm{sn}^2 z - Q$, satisfying the equation

$$\frac{d^2w}{dz^2} = (a_0+6k^2\,\mathrm{sn}^2z)w.$$

It is easily seen that

$$3k^2Q^2-2(1+k^2)Q+1 = 0, \qquad a_0\,Q = -2.$$

We assume the formal solution

$$u\,\mathrm{el}_2^0(z) = uE_2^0(z)+\sum q^t f_t(z),$$
$$a = a_0+\sum c_t q^t, \qquad b = 6+\sum d_t q^t,$$

and substitute in (1). Reducing and comparing coefficients of powers of q yields the equations

$$f_1''-(a_0+6k^2\,\mathrm{sn}^2z)f_1 = (c_1+d_1\,k^2\,\mathrm{sn}^2z+k^4\,\mathrm{sn}^4z)uE_2^0(z), \tag{9}$$

$$f_2''-(a_0+6k^2\,\mathrm{sn}^2z)f_2 = (c_1+d_1\,k^2\,\mathrm{sn}^2z+k^4\,\mathrm{sn}^4z)f_1+(c_2+d_2\,k^2\,\mathrm{sn}^2z)uE_2^0(z),$$
$$\text{etc.} \tag{10}$$

When the right-hand side is expanded, (9) becomes

$$f_1''-(a_0+6k^2\,\mathrm{sn}^2z)f_1$$
$$= -c_1\,Q+(c_1-d_1\,k^2Q)\mathrm{sn}^2z+k^2(d_1-k^2Q)\mathrm{sn}^4z+k^4\,\mathrm{sn}^6z. \tag{11}$$

Now the normalization condition implies that, in this case, the coefficient of sn^2z in $u\,\mathrm{el}_2^0(z)$ must be unity for all values of q, and hence f_1, f_2, etc. will not contain any term in sn^2z. We therefore assume a particular integral of (11) in the form $f_1 = A\,\mathrm{sn}^4z+B$. Substituting and comparing coefficients of powers of $\mathrm{sn}\,z$, we have

$$14k^2A = k^4, \qquad \{a_0+16(1+k^2)\}A = k^2(k^2Q-d_1),$$
$$12A-6k^2B = c_1-d_1\,k^2Q, \qquad a_0B = c_1\,Q.$$

Solving these in succession and substituting for Q and a_0, we have

$$A = \tfrac{1}{14}k^2, \qquad d_1 = -\tfrac{2}{3}(1+k^2)+\tfrac{4}{21}\lambda, \qquad c_1 = \frac{-k^2(2+2k^2-7\lambda)}{21\lambda},$$

$$B = \frac{(1+k^2)(-10+16k^2-10k^4)+\lambda(-10+k^2-10k^4)}{2.3^3.7k^2\lambda}.$$

Thus we have a particular integral of (9). The complementary function is $\alpha E_2^0(z)+\beta F_2^0(z)$, where $F_2^0(z)$ is the standard second solution of Lamé's equation corresponding to $E_2^0(z)$. Since f_1 is to be doubly-periodic, and also to contain no term in $\mathrm{sn}^2 z$, we reject this complementary function completely and have $f_1 = A\,\mathrm{sn}^4 z+B$, where A and B have the values given above.

The expressions already obtained for f_1, c_1, d_1 may now be substituted in (10) and the same process continued. The work is obviously very complex, and it may be doubted whether the practical value of general formulae are worth the very heavy algebraic manipulation involved. If, however, the numerical value of k^2 is known, the work becomes comparatively simple since λ, Q, a_0 can at once be given numerical values also.

It is worth noting, however, that $u\,\mathrm{el}_2^1(z)$, $a^{(u)1}_{\ 2}$, $b^{(u)1}_{\ 2}$, can be obtained from $u\,\mathrm{el}_2^0(z)$, $a^{(u)0}_{\ 2}$, $b^{(u)0}_{\ 2}$, merely by changing the sign of λ.

5. Checking

The solutions obtained above may, of course, be fully checked by substituting the series for $\mathrm{el}(z)$, a, b in the equation (1). This, however, is a lengthy process, and a much simpler check is provided, at least for the expressions for a and b, by means of certain transformation formulae obtained by the author on lines similar to Erdélyi's work on the corresponding formulae for Lamé functions [(6) § 5].

If, in the expressions for a, b, we write $-q$ for q and k'^2 for k^2 (where $k^2+k'^2 = 1$), I shall denote the results by $\bar{a}$, $\bar{b}$, respectively. Then the transformation formulae are

$$\bar{b}^{(u)m}_{2n}-b^{(u)n-m}_{2n} = 2q, \qquad \bar{a}^{(u)m}_{2n}+\bar{b}^{(u)m}_{2n}+a^{(u)n-m}_{2n} = q, \tag{12a}$$

$$\bar{b}^{(s,d)m}_{2n+1}-b^{(d,s)n-m}_{2n+1} = 2q, \qquad \bar{a}^{(s,d)m}_{2n+1}+\bar{b}^{(s,d)m}_{2n+1}+a^{(d,s)n-m}_{2n+1} = q, \tag{12b}$$

$$\bar{b}^{(c)m}_{2n+1}-b^{(c)n-m}_{2n+1} = 2q, \qquad \bar{a}^{(c)m}_{2n+1}+\bar{b}^{(c)m}_{2n+1}+a^{(c)n-m}_{2n+1} = q, \tag{12c}$$

$$\bar{b}^{(sc,cd)m}_{2n+2}-b^{(cd,sc)n-m}_{2n+2} = 2q, \qquad \bar{a}^{(sc,cd)m}_{2n+2}+\bar{b}^{(sc,cd)m}_{2n+2}+a^{(cd,sc)n-m}_{2n+2} = q, \tag{12d}$$

$$\bar{b}^{(sd)m}_{2n+2}-b^{(sd)n-m}_{2n+2} = 2q, \qquad \bar{a}^{(sd)m}_{2n+2}+\bar{b}^{(sd)m}_{2n+2}+a^{(sd)n-m}_{2n+2} = q, \tag{12e}$$

$$\bar{b}^{(scd)m}_{2n+3}-b^{(scd)n-m}_{2n+3} = 2q, \qquad \bar{a}^{(scd)m}_{2n+3}+\bar{b}^{(scd)m}_{2n+3}+a^{(scd)n-m}_{2n+3} = q. \tag{12f}$$

In (12 b) and (12 d), either the first, or the second, character must be taken throughout. As an example of the use of these formulae, let us take (12 b) and put $n = m = 0$. Taking the first character gives

$$\bar{b}^{(s)0}_{\ 1}-b^{(d)0}_{\ 1} = 2q, \qquad \bar{a}^{(s)0}_{\ 1}+\bar{b}^{(s)0}_{\ 1}+a^{(d)0}_{\ 1} = q. \tag{13}$$

Putting $-q$ for q and $1-k^2$ for k^2 in the formulae (2) of Table III, we have

$$\bar{a}^{(s)0}_{1} = -2+k^2-q\frac{(3-3k^2)}{5}+q^2\frac{(6-9k^2+3k^4)}{875},$$

$$\bar{b}^{(s)0}_{1} = 2+q\frac{(8-4k^2)}{5}+q^2\frac{2(3-3k^2+2k^4)}{875};$$

and it is now a simple matter to verify that both the equations (13) are satisfied.

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THE BEHAVIOUR OF INTEGRAL FUNCTIONS DETERMINED FROM THEIR TAYLOR SERIES

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1. In this paper I shall prove the following theorem.

Theorem 1. *Let* $f(z) = \sum_{1}^{\infty} a_n z^n$ *be an integral function and* α *be a positive number. If* $\left(\frac{n+1}{n}\right)^{\alpha}\left|\frac{a_n}{a_{n+1}}\right|$ *is ultimately a steadily increasing function of* n, *then*

$$M(r,f) < \{1+o(1)\}\alpha^{-\alpha-1}\Gamma(1+\alpha)e^{\alpha}\nu(r,f)\mu(r,f). \tag{1}$$

Theorem 2. *If* $f(z)$ *satisfies the conditions of Theorem* 1, *then*

$$\limsup_{r\to\infty}\frac{\log M(r,f)}{\log \mu(r,f)} \leqslant 1+\alpha^{-1}. \tag{2}$$

The above theorems are easily proved and their value depends on their being best-possible. This is shown by

Theorem 3. *Given* $\alpha > 0$, *there exists an integral function*

$$g_{\alpha}(z) = \sum_{1}^{\infty} b_n z^n$$

such that $\left(\frac{n+1}{n}\right)^{\alpha}\left|\frac{b_n}{b_{n+1}}\right|$ *is a steadily increasing function of* n *and*

$$M(r,g_{\alpha}) > \{1-o(1)\}\alpha^{-\alpha-1}\Gamma(1+\alpha)e^{\alpha}\nu(r,g_{\alpha})\mu(r,g_{\alpha}) \tag{3}$$

for an infinite sequence of r, *and*

$$\limsup_{r\to\infty}\frac{\log M(r,f)}{\log \mu(r,f)} = 1+\alpha^{-1}. \tag{4}$$

From the construction of $g_{\alpha}(z)$ of Theorem 3 it will be seen how an integral function $h(z) = \sum_{1}^{\infty} c_n z^n$ can be found such that $\left|\frac{c_n}{c_{n+1}}\right|$ steadily increases with n and

$$\limsup_{r\to\infty}\frac{M(r,h)}{\{\nu(r,k)\}^{k}\mu(r,h)} = \infty,$$

$$\limsup_{r\to\infty}\frac{\log_k M(r,h)}{\mu(r,h)} = \infty,$$

where k is any positive integer, however large. Hence there can be no general results corresponding to Theorems 1 and 2 for the case $\alpha = 0$.

At the end of the paper I shall use Theorem 2 to extend a result due to Singh and Shah (**1**).

2. The proofs of Theorems 1 and 2 depend on an attenuated form of the Wiman–Valiron method [(**3**) 93]. If $f(z)$ satisfies the conditions of Theorem 1, then

$$\phi_\alpha(z) = \sum_1^\infty \frac{a_n}{n^\alpha} z^n$$

is an integral function for which all integers beyond some definite integer are central indices. Let $\nu = \nu(r, \phi_\alpha)$ and then

$$\frac{|a_n|}{n^\alpha} r^n \leqslant \frac{|a_\nu|}{\nu^\alpha} r^\nu \quad (n = 1, 2, \ldots),$$

so that
$$\frac{|a_n| r^n}{|a_\nu| r^\nu} \leqslant \frac{n^\alpha}{\nu^\alpha} \quad (n = 1, 2, \ldots).$$

For any ν, R (< 1) can be chosen so that, for all n, $n^\alpha R^n \leqslant \nu^\alpha R^\nu$, and hence
$$\frac{|a_n|(rR)^n}{|a_\nu|(rR)^\nu} \leqslant \frac{n^\alpha R^n}{\nu^\alpha R^\nu} \leqslant 1, \tag{5}$$

which gives
$$\nu(r, \phi_\alpha) = \nu(R, F_\alpha) = \nu(rR, f), \tag{6}$$

where $F_\alpha(z) = \sum\limits_1^\infty n^\alpha z^n$. Further,

$$\begin{aligned} \mu(rR, f) &= |a_\nu|(rR)^\nu \\ &= \frac{|a_\nu|}{\nu^\alpha} r^\nu . \nu^\alpha R^\nu \\ &= \mu(r, \phi_\alpha)\mu(R, F_\alpha). \end{aligned} \tag{7}$$

Since all large integers in turn become central indices of both $\phi_\alpha(z)$ and $F_\alpha(z)$, it follows that the values rR include all numbers exceeding some definite bound.

From (5) we get

$$\begin{aligned} \frac{\sum\limits_1^\infty |a_n|(rR)^n}{|a_\nu|(rR)^\nu} &\leqslant \frac{\sum\limits_1^\infty n^\alpha R^n}{\nu^\alpha R^\nu} \\ &\sim \frac{\Gamma(1+\alpha)(1-R)^{-\alpha-1}}{\nu^\alpha R^\nu} \end{aligned} \tag{8}$$

as $R \to 1-0$ [(2) 225]. Now

$$\frac{d}{dx}(\alpha \log x + x \log R) = \frac{\alpha}{x} + \log R,$$

so that $x^\alpha R^x$ increases steadily for $x > 0$ till

$$x = \xi = -\alpha/\log R$$

and then decreases steadily. Hence, if $\nu = \nu(R, F_\alpha)$, then $|\nu - \xi| \leqslant 1$. Therefore

$$\frac{\Gamma(1+\alpha)(1-R)^{-\alpha-1}}{\nu^\alpha R^\nu} < \frac{\Gamma(1+\alpha)}{\nu^\alpha}\exp\left(\frac{\alpha\nu}{\nu-1}\right)\left\{1-\exp\left(-\frac{\alpha}{\nu+1}\right)\right\}^{-\alpha-1}$$

$$\sim \alpha^{-\alpha-1}\Gamma(1+\alpha)e^\alpha . \nu, \tag{9}$$

as $\nu \to \infty$. Since, as has already been observed, all large values are assumed by rR, we get (1) from (6), (8), (9).

Also, from (1),

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{\log \mu(r,f)} \leqslant 1 + \limsup_{r\to\infty} \frac{\log \nu(r,f)}{\log \mu(r,f)}$$

and, using (6) and (7), we see that the limit on the right-hand side above does not exceed

$$\limsup_{R\to 1} \frac{\log \nu(R,F_\alpha)}{\log \mu(R,F_\alpha)} \leqslant \limsup_{\nu\to\infty} \frac{\log \nu}{\alpha \log \nu + \alpha\nu/(\nu-1)} = \frac{1}{\alpha},$$

which gives (2).

This completes the proof of Theorems 1 and 2.

3. We shall now construct the function $g_\alpha(z)$ of Theorem 3. Let $r_1 < r_2 < \dots < r_k$ be k positive numbers. Define

$$r_n = r_k \exp\{k^{-5} + \dots + (n-1)^{-5}\}$$

for $k < n \leqslant p_1^2$, where p_1 is chosen so that

$$k < p_1^{\frac{1}{4}}, \qquad \frac{r_k^{k-1}\exp\frac{4}{3}k^{-3}}{r_1 \dots r_{k-1}} < \log p_1.$$

Then we get

$$\frac{r_{p_1}^{p_1}}{r_1 \dots r_{p_1}} = \frac{r_k^{k-1}}{r_1 \dots r_{k-1}} \exp\{k^{-4} + \dots + (p_1-1)^{-4}\} < \frac{r_k^{k-1}\exp(\frac{4}{3}k^{-3})}{r_1 \dots r_{k-1}}$$

$$< \log p_1.$$

When $p_1 < n \leqslant p_1^2$,

$$\frac{r_{p_1}^n}{r_1 \ldots r_n} \bigg/ \frac{r_{p_1}^{p_1}}{r_1 \ldots r_{p_1}} = \frac{r_{p_1}^{n-p_1}}{r_{p_1+1} \ldots r_n}$$

$$= \exp\left[-\left\{\frac{n-p_1}{p_1^5} + \frac{n-p_1+1}{(p_1+1)^5} + \ldots + \frac{1}{(n-1)^5}\right\}\right]$$

$$> \exp\left\{-\frac{(n-p_1)^2}{p_1^5}\right\}$$

$$> \exp(-p_1^{-1}),$$

and, when $p_1^{\frac{1}{2}} \leqslant n < p_1$,

$$\frac{r_{p_1}^n}{r_1 \ldots r_n} \bigg/ \frac{r_{p_1}^{p_1}}{r_1 \ldots r_{p_1}} = \frac{r_{n+1} \ldots r_{p_1}}{r_{p_1}^{p_1-n}}$$

$$= \exp\left[-\left\{\frac{p_1-n-1}{(p_1-1)^5} + \frac{p_1-n-2}{(p_1-2)^5} + \ldots + \frac{1}{(n+1)^5}\right\}\right]$$

$$> \exp\left\{-\frac{(p_1-n-1)^2}{(n+1)^5}\right\}$$

$$> \exp(-p_1^{-\frac{1}{2}}).$$

Thus, if $p_1^{\frac{1}{2}} \leqslant n \leqslant p_1^2$,

$$\frac{r_{p_1}^n}{r_1 \ldots r_n} > \exp(-p_1^{-\frac{1}{2}}) \frac{r_{p_1}^{p_1}}{r_1 \ldots r_{p_1}}.$$

Now define $r_{p_1^2+1} = r_{p_1^2} + 1$ and proceed as before with k replaced by p_1^2+1. Then we obtain an integer $p_2 > p_1$ such that

$$\frac{r_{p_2}^{p_2}}{r_1 \ldots r_{p_2}} < \log p_2$$

and, when $p_2^{\frac{1}{2}} \leqslant n \leqslant p_2^2$,

$$\frac{r_{p_2}^n}{r_1 \ldots r_n} > \exp(-p_2^{-\frac{1}{2}}) \frac{r_{p_2}^{p_2}}{r_1 \ldots r_{p_2}}.$$

Proceeding step by step in this way we construct a sequence of positive numbers $r_1, \ldots, r_n, \ldots$ such that r_n increases steadily to infinity with n. For an infinite sequence of integers $p_1, \ldots, p_n, \ldots$

$$\frac{r_{p_n}^{p_n}}{r_1 \ldots r_{p_n}} < \log p_n \tag{10}$$

and, for $p_n^{\frac{1}{2}} \leqslant m \leqslant p_n^2$,

$$\frac{r_{p_n}^m}{r_1 \dots r_m} > \exp(-p_n^{-\frac{1}{2}}) \frac{r_{p_n}^{p_n}}{r_1 \dots r_{p_n}}. \tag{11}$$

We now put
$$g_\alpha(z) = \sum_1^\infty \frac{n^\alpha z^n}{r_1 \dots r_n}.$$

Since r_n increases steadily to infinity with n, it follows that $g_\alpha(z)$ is an integral function satisfying the conditions of Theorem 1. From the proof of Theorem 1 we have $\nu(r_{p_n} R_n, g_\alpha) = p_n$, where

$$\exp\left(-\frac{\alpha}{p_n - 1}\right) \leqslant R_n \leqslant \exp\left(-\frac{\alpha}{p_n + 1}\right). \tag{12}$$

From (11),

$$g_\alpha(r_{p_n} R_n) > \sum_{p_n^{\frac{1}{2}}}^{p_n^2} \frac{m^\alpha (r_{p_n} R_n)^m}{r_1 \dots r_m} > \frac{\exp(-p_n^{-\frac{1}{2}}) r_{p_n}^{p_n}}{r_1 \dots r_{p_n}} \sum_{p_n^{\frac{1}{2}}}^{p_n^2} m^\alpha R_n^m. \tag{13}$$

Also

$$\sum_{p_n^{\frac{1}{2}}}^{p_n^2} m^\alpha R_n^m > \{1 - \epsilon(p_n)\} \Gamma(1+\alpha)(1-R)^{-\alpha-1} - \left(\sum_1^{p_n^{\frac{1}{2}}} + \sum_{p_n^2}^\infty\right) m^\alpha R_n^m, \tag{14}$$

where $\epsilon(p_n) \to 0$ as $p_n \to \infty$. For the sums on the right-hand side of (14) we get

$$\begin{aligned} \sum_1^{p_n^{\frac{1}{2}}} m^\alpha R_n^m &< \frac{p_n^{\alpha/2}}{1 - R_n} \\ &\sim \alpha^{-1} p_n^{\alpha/2+1} \end{aligned} \tag{15}$$

as $p_n \to \infty$ from (12), and

$$\begin{aligned} \sum_{p_n^2}^\infty m^\alpha R_n^m &= \sum_{p_n^2}^\infty (m^{\alpha/m} R_n)^m \\ &< \sum_{p_n^2}^\infty R_n'^m, \end{aligned}$$

where, from (12),

$$R_n' \leqslant \exp\left(-\frac{\alpha}{p_n+1} + \frac{2\alpha \log p_n}{p_n^2}\right) < \exp\left(-\frac{\alpha}{2ep_n}\right), \tag{16}$$

provided that $p_n > (2e-5)^{-1}$. Therefore, if $p_n > (2e-5)^{-1}$, then

$$\begin{aligned} \sum_{p_n^2}^\infty m^\alpha R_n^m &< \frac{R_n'^{p_n^2}}{1 - R_n'} \\ &< \{1 + o(1)\} 2\alpha^{-1} e p_n \exp\left(-\frac{\alpha p_n}{2e}\right), \end{aligned} \tag{17}$$

from (16). Since, from (12),

$$(1 - R_n)^{-\alpha-1} \sim \alpha^{-\alpha-1} \Gamma(1+\alpha) p_n^{1+\alpha}$$

as $p_n \to \infty$, it follows that

$$g_\alpha(r_{p_n} R_n) > \{1-o(1)\}\alpha^{-\alpha-1}\Gamma(1+\alpha)p_n^{1+\alpha}\frac{r_{p_n}^{p_n}}{r_1 \dots r_{p_n}}, \tag{18}$$

from (13), (15), (17). Now

$$\mu(r_{p_n} R_n, g_\alpha) = \frac{p_n^\alpha (r_{p_n} R_n)^{p_n}}{r_1 \dots r_{p_n}} \tag{19}$$

and, since $R_n^{p_n} \sim e^{-\alpha}$ as $p_n \to \infty$, we get, from (18) and (19),

$$g_\alpha(r) > \{1-o(1)\}\alpha^{-\alpha-1}\Gamma(1+\alpha)e^\alpha \nu(r, g_\alpha)\mu(r, g_\alpha)$$

as $r \to \infty$ through the sequence $r_{p_n} R_n$. This proves (3).

From (19) we have, using (10) and (12),

$$\mu(r_{p_n} R_n, g_\alpha) < p_n^\alpha \exp\left(-\frac{\alpha p_n}{p_n+1}\right)\log p_n,$$

and so

$$\limsup_{r_{p_n} R_n \to \infty} \frac{\log p_n}{\log \mu(r_{p_n} R_n, g_\alpha)} \geqslant \frac{1}{\alpha}. \tag{20}$$

Since $p_n = \nu(r_{p_n} R_n, g_\alpha)$, we get (4) from (19), (20) and Theorem 2.

This completes the proof of Theorem 3.

4. I shall now show how Theorem 2 can be used to generalize a result given by Shah and Singh (**1**). These authors have proved the following theorem.

Let $\theta(x)$ satisfy the following conditions:

(i) *$\theta(x)$ is positive and non-decreasing for $x \geqslant x_0$ and tends to infinity with x,*

(ii)
$$I(x) = \int_{x_0}^{x} \frac{dt}{t\theta(t)},$$

tends to infinity with x,

(iii)
$$\frac{x\theta'(x)}{\theta(x)} \leqslant c < 1$$

for $x \geqslant x_0$. Let N be an integer such that $I(N) \geqslant 1$. Then

$$F(z) = \sum_{N}^{\infty} \{z/I(n)\}^n$$

is an integral function of infinite order such that

$$\lim_{r\to\infty} \frac{\log \mu(r, F)\theta\{\log \mu(r, F)\}}{\nu(r, F)} = 0. \tag{21}$$

If $\theta(x)$ also satisfies the condition

(iv)
$$I(n^p) - I(n) > \frac{2}{\theta(x+1)},$$

where p is some (fixed) integer for all large n, then

$$\lim_{r\to\infty} \frac{\log M(r,F)\theta\{\log M(r,F)\}}{\nu(r,F)} = 0. \tag{22}$$

Shah and Singh showed that (22) follows from (21) and (i)–(iii) if $\log M(r,F) \sim \log \mu(r,F)$. To prove this last result they made use of (iv). However, I shall show that $\log M(r,F) \sim \log\mu(r,F)$ follows from (i)–(iii) alone.

Let

$$\chi_\lambda(x) = \lambda \log\left(\frac{x+1}{x}\right) + (x+1)\log I(x+1) - x\log I(x).$$

Then, using (ii), we have

$$\chi'_\lambda(x) = -\frac{\lambda}{x(x+1)} + \frac{1}{\theta(x+1)I(x+1)} + \log I(x+1) - \frac{1}{\theta(x)I(x)} - \log I(x).$$

From the mean-value theorem we get

$$\log I(x+1) - \log I(x) > \frac{1}{(x+1)\theta(x+1)I(x+1)},$$

and, using (iii),

$$\frac{1}{\theta(x+1)I(x+1)} - \frac{1}{\theta(x)I(x)} > -\frac{c}{x\theta(x)I(x)} - o\left(\frac{1}{x^2}\right)$$

and

$$\frac{1}{(x+1)\theta(x+1)I(x+1)} - \frac{1}{x\theta(x)I(x)} > -o\left(\frac{1}{x^2}\right).$$

From these results we find that

$$\chi'_\lambda(x) > -\frac{\lambda}{x(x+1)} - o\left(\frac{1}{x^2}\right) + \frac{1-c}{x\theta(x)I(x)}.$$

Condition (iii) gives $\theta(x) < \{1+o(1)\}x^c$ and, since $I(x) = o(\log x)$, it follows that

$$\chi'_\lambda(x) > -\frac{\lambda}{x(x+1)} - o\left(\frac{1}{x^2}\right) + \frac{\{1-o(1)\}(1-c)}{x^{1+c}\log x},$$

which is ultimately greater than zero for any λ. Hence $\chi_\lambda(x)$ is ultimately a steadily increasing function of x, and so

$$\exp\{\chi_\lambda(n)\} = \left(\frac{n+1}{n}\right)^\lambda \frac{\{I(n+1)\}^{n+1}}{\{I(n)\}^n}$$

is ultimately a steadily increasing function of n for any λ, however large. From Theorem 2 we get

$$\log M(r, F) \sim \log \mu(r, F).$$

This shows that condition (iv) is superfluous.

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A CLASS OF MAJORITY GAMES

By J. R. ISBELL (*Washington, D.C.*)

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THIS paper is concerned with the description of a class of combinatorial objects, the weighted majority games of von Neumann and Morgenstern (2). I find a rather full description of a certain subclass containing 2^{n-4} n-person games for each n not less than 4. This class is probably a negligibly small part of the whole; but the corollary that there are at least 2^{n-4} different n-person weighted majority games is better than any previously known lower bound. (For six players there are exactly 14 such games, and for seven at least 110.) Beyond our little subclass we have a few remarks; but it is not even known whether in general there is a unique natural choice of weights. The *homogeneous* weights are unique, when they exist (1); and I show that such weights cannot increase faster than the Fibonacci numbers.

A (strong, weighted) *majority game* is a game definable by weights w_i assigned to the players; the set S of players wins if and only if its weight $w(S) = \sum_{i \in S} w_i$ is greater than the weight of its complement $N-S$. The essential idea consists in the specification of *winning sets* (and their complements, the *losing sets*). There can be no ties. Thus we deal with n-tuples of non-negative real numbers, organized into equivalence classes which are essentially open convex cones. I shall suppose that no dummy players are present, so that each player i is in at least one minimal winning set.

Given an abstract game, i.e. a specification of winning sets, one can determine whether it is a majority game by constructive methods: it is a question of solvability of a system of strict linear inequalities. However, no simple constructive characterization is known. Thus we face the questions, 'What pairs of n-tuples of weights are equivalent?' and 'What properties have the equivalence classes?' Since I have no answers, I shall minimize the discussion of these questions. But obviously open cones contain lattice points, i.e. there are integer weights for any majority game; and minimal integer weights are easily found. In all known examples there exist minimum integer weights, i.e. the cones in question have the very unusual property of containing minimum lattice points. Any substantial contribution to the theory of weighted majority games ought to determine whether this property is accidental or general.

For each player i in a minimal winning set S, there is a minimal winning set T whose intersection with S consists just of i. This is well known and follows from the fact that $(N-S) \cup \{i\}$ wins but $N-S$ does not. Several corollaries follow directly.

Let us write S' for $N-S$. Note the identity

$$w(S \cap T) = w(S' \cap T') + \tfrac{1}{2}[w(S) - w(S') + w(T) - w(T')]. \qquad (1)$$

When S and T are winning sets with intersection $\{i\}$, this means that $w_i > w(S' \cap T')$.

[1] *Each pair of players lies in a minimal winning set.*

Proof. Let the players be i and j ($w_i \geqslant w_j$). Let S and T be winning sets meeting in $\{j\}$; since $w_j > w(S' \cap T')$, i is not in $S' \cap T'$ and hence is in S or in T.

[2] *If each player is the intersection of two sets S such that*

$$w(S) = w(S') + 1,$$

then the weights w_i are minimum integer weights.

Proof. Let w_i be the smallest non-integer weight. From (1) there is a set $S' \cap T'$ of weight $w_i - 1$, which is absurd. Then the weights are integers; suppose that they are not minimum. Let (v_i) be integer weights and w_j the smallest weight such that $v_j < w_j$. From (1) there is a set $S' \cap T'$ such that $w(S' \cap T') = w_j - 1$. By choice of j,

$$v_j \leqslant w_j - 1, \qquad v(S' \cap T') \geqslant w(S' \cap T').$$

Substitution gives $$v_j \leqslant v(S' \cap T'),$$

a contradiction.

The weights are said to be *homogeneous* when $w(S) - w(S')$ is the same for all minimal winning sets S (**2**); letting this common difference be unity, we see that

[3] *Homogeneous weights are minimum integer weights.*

[4] *Homogeneous weights are unique.*

Corollary 4 has previously been established by an argument on the rank of the incidence matrix of the game (**1**).

From the hypothesis of Corollary 2 we have deduced that for each i there is a set of weight $w_i - 1$. A simple induction proves that

[5] *Every integer from 0 to $w(N)$ is $w(S)$ for some S.*

With homogeneous weights we can use Corollary 1 to set aside the player j whose weight is the largest of those less than w_i: that is, i and j

are in a common minimal winning set S and hence there is $S' \cap T'$ excluding j such that $w(S' \cap T') = w_i - 1$. This leads to

[6] *The k-th from least weight of homogeneous weights is at most the k-th Fibonacci number.*

Proof. Recall the definition of the Fibonacci numbers:

$$a_1 = a_2 = 1, \qquad a_n = a_{n-1} + a_{n-2}.$$

By induction, the sum of the first n Fibonacci numbers is $a_{n+2} - 1$. Now the least weight w_i is 1; and the kth from least exceeds the sum of the smallest $k-2$ by at most 1. Hence the corollary.

With Corollaries 1, 5, 6 (and 3), one has the beginning of a description of those sets of integers which are homogeneous weights. Corollary 6 is sharp; letting the first $n-2$ weights be the corresponding Fibonacci numbers,

$$w_{n-1} = a_{n-2}, \qquad w_n = a_{n-1},$$

one has homogeneous weights for each $n \geqslant 3$.

We turn now to the narrow class C of all majority games with n ($\geqslant 3$) non-dummy players and n minimal winning sets. From (1) we know (a) that this is the least possible number of minimal winning sets, (b) that each game in C has a main solution, (c) and therefore that each game in C has homogeneous weights. Further, we assert the next three paragraphs.

The class C consists of the three-person majority game (weights are 1, 1, 1) and 2^{n-4} games of n persons for each $n \geqslant 4$. For $n \geqslant 4$, arranging the homogeneous weights w_i in non-increasing order, we have

$$w_1 > w_2 = w_3, \qquad w_{n-1} = w_n.$$

This leaves $n-4$ choices,

$$w_i > w_{i+1} \quad \text{or} \quad w_i = w_{i+1},$$

which may be made freely. The n players are then partitioned into h equivalence classes $E_1,\dots, E_h$, arranged in order of decreasing w_i. One minimal winning set S_1 consists of the union of all E_j such that $j \equiv h \pmod 2$. The other minimal winning sets S_i may be indexed by the players $i > 1$. For i in E_j, S_i consists of i together with all E_k such that $k < j$ and $k \equiv j+1 \pmod 2$.

The transpose of the incidence matrix of players upon minimal winning sets for a game G in C is the corresponding incidence matrix for another game G^* in C. The three-person and four-person games are self-dual; for $3 \leqslant i \leqslant n-2$, $w_i > w_{i+1}$ in G if and only if $w_{n-i+1} > w_{n-i+2}$ in G^*.

The minimal winning set S_1 corresponds to the player 1 in G^*; the other S_i correspond to $n+2-i$.

For i in E_h, $w_i = 1$. For i in E_j such that

$$1 < j \equiv h+1 \pmod 2,$$

w_i is the sum of the weights of all players in E_k $(k > j;\ k \equiv h \pmod 2)$. If $1 < j \equiv h \pmod 2$, then w_i is the sum of the weights of all players in $E_k(k > j;\ k \equiv h+1 \pmod 2)$ plus 1. Finally, w_1 is the sum of all $w_i\,(i \geqslant 4)$ plus 1.

The reader may verify that we have correctly described 2^{n-4} games in C for each $n \geqslant 4$. Thus our assertions become justified when we show that there are at most 2^{n-4} games of n persons in C. We do this by an induction, reducing n-person games to $(n-1)$-person games and showing that the transformation is at most two-to-one. For the initial cases $n = 3$ and $n = 4$, we refer to (**2**). We remark that besides the involution $G \Leftrightarrow G^*$ and the reduction just mentioned, there are at least two other natural transformations in C. One consists of deleting all players in E_h, i.e. of weight 1; the remaining weights are of course not homogeneous but they define a game in C if there are three players left. Another transformation gives an h-person game (if $h \geqslant 3$), by aggregating each equivalence class into a player having the total weight of the class.

Our argument begins with a fifth transformation; but this is only a hypothetical construction leading to a proof by contradiction.

[I] *For G in C with $n \geqslant 4$ players, each minimal winning set contains none, one, or all of the players of weights* 1.

Proof. Suppose [I] is false in G. Let w be the homogeneous weights for G, and define G' by deleting two players, i, j, of weight 1. The sum of the weights in G' is odd and hence G' is a strong majority game with $n-2$ players. It contains players of weight 1 (or else [I] holds); and, since [I] is false, they are not dummies. Hence G' has no dummies and therefore it has at least $n-2$ minimal winning sets (**1**). Now consider the $t\ (\geqslant n-2)$ partitions of the players of G' into a minimal winning set S and its complement T. Either $w(S) = w(T)+3$ or $w(S) = w(T)+1$. In the first case S is a minimal winning set in G containing neither i nor j. In the second case

$$S \cup \{i\}, \quad S \cup \{j\}, \quad T \cup \{ij\}$$

are three different minimal winning sets of G. Since G has only n minimal winning sets, the second case cannot arise more than once. But from $n \geqslant 4$ and the denial of [I], one easily sees that this is a contradiction. Therefore [I] holds.

[II] *Each player of weight* 1 *is in just two minimal winning sets.*

Proof. We may begin by noting that [II] holds in the unique example for $n < 4$. Now assume $n \geqslant 4$. Delete one player i of weight 1. There remains a set of players with integral weights whose sum is even; but from [I] we know that, if this set is partitioned into two complementary sets of equal weight, then all players of weight 1 are in one element of the partition. Thus by replacing each weight 1 by $1+\epsilon$, for sufficiently small positive ϵ, we obtain a majority game $R(G)$ with $n-1$ players who are obviously not dummies. As before, $R(G)$ has at least $n-1$ minimal winning sets. Now the minimal winning sets of G which contain i occur in pairs, S, $S' \cup \{i\}$. For each such pair, exactly one of S and S' wins in $R(G)$. All other minimal winning sets of $R(G)$ win in G, and therefore, (since G has only one more minimal winning set than $R(G)$), i is in only two minimal winning sets, as asserted.

Now [II] is proved. Note that $R(G)$ must have just $n-1$ minimal winning sets; therefore $R(G)$ has homogeneous weights, and, if we substitute these for the weights just constructed, the reduction can be iterated.

How is G obtained from $R(G)$? At the risk of boring the reader, I emphasize that the weights given for $R(G)$ in my construction are not given with the game. What we do know is that some minimal winning set S of $R(G)$ does not win in G; an extra player i is added, and $S \cup \{i\}$ becomes a minimal winning set. One property of S which we may deduce from the weights for G is that, if any player j not in S is added to S, then the result $S \cup \{j\}$ contains at least one minimal winning set different from S. Now $R(G)$ is a homogeneous weighted majority game; and this property for its minimal winning sets is easily seen to be equivalent to containing at least one player of weight 1. Since $R(G)$ is in C, we know from [I], [II], and the substitutability of players having the same weight that there are only two such sets inequivalent under motions of the game. Therefore, there are at most two n-person games G in C having the same $R(G)$; and there are at most 2^{n-4} n-person games in C, for $n \geqslant 4$.

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'ALMOST CONVERGENCE' AND UNIFORMLY DISTRIBUTED SEQUENCES

By G. M. PETERSEN (*Swansea*)

[Received 31 January 1956]

1. I SHALL describe a method of summation α by a rectangular array of linear transformations of a sequence $\{s_i\}$;

$$\begin{array}{cccccccccc} t_1^{(1)} & t_2^{(1)} & . & . & . & t_n^{(1)} & . & . & . \\ t_1^{(2)} & t_2^{(2)} & . & . & . & t_n^{(2)} & . & . & . \\ . & . & . & . & . & . & . & . & . \\ t_1^{(m)} & t_2^{(m)} & . & . & . & t_n^{(m)} & . & . & . \\ . & . & . & . & . & . & . & . & . \end{array}$$

where $t_n^{(m)} = \sum_{i=1}^{\infty} a_{ni}^{(m)} s_i$. I shall say that a sequence $\{s_i\}$ is *α-summable* if the transformations $t_n^{(m)}$ converge uniformly in (m) to a limit. One method of summation of this type is the method of 'almost convergence' [Lorentz (**1**)]. If we let

$$t_n^{(m)} = m^{-1}(s_n+s_{n+1}+...+s_{n+m})$$

we have 'almost convergence' described by the uniform convergence of the rows of the corresponding array. It is also evident that any matrix method can be described as a method of this type. If $A = (a_{mi})$ and $t_m = \sum_{i=1}^{\infty} a_{mi} s_i$, we need only take $t_n^{(m)} = t_m$ for all n.

Associated with each method of summation α is the family $\mathfrak{A}$ of all possible matrices that can be developed by selecting the first row of the matrix from the first row of the array, the second from the second row of the array and so on. This leads to the first theorem.

THEOREM 1. *The method α sums exactly those sequences summed by every member of $\mathfrak{A}$.*

Proof. If a sequence is not limited by some member of $\mathfrak{A}$, then it is evident that the rows cannot converge uniformly. Hence α is certainly contained by all the members of $\mathfrak{A}$.

On the other hand, suppose that a sequence $\{s_k\}$ is limited by all the members of $\mathfrak{A}$ to s but not by α. Since it is not α-limitable, there is an infinite set $\{m_\mu\}$ and a corresponding $\{n(m_\mu)\}$ such that

$$|t_{n(m_\mu)}^{(m_\mu)} - s| > \epsilon$$

for some ϵ. This means that the matrix which has an infinite number of rows selected from

$$\{t^{(m_\mu)}_{n(m_\mu)}\},$$

and an infinite number of rows from some matrix that limits $\{s_k\}$ will be a member of $\mathfrak{A}$ that does not limit $\{s_k\}$. This contradiction proves the assertion.

We see incidentally that α is regular if and only if all the members of $\mathfrak{A}$ are regular. The method can be strongly regular in the sense of Lorentz (**1**) if and only if all the members of $\mathfrak{A}$ are strongly regular.

2. We suppose that $0 \leqslant s_n \leqslant 1$ for every n, and denote the interval $0 \leqslant a \leqslant x \leqslant b \leqslant 1$ by I. We denote by $I(x)$ the characteristic function of I which is 1 in I and 0 elsewhere.

A sequence $\{s_k\}$ is then said to be *uniformly distributed* if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} I(s_k) = b-a$$

for every I.

A well-known theorem due to Weyl (**2**) states that, if $\{s_k\}$ is uniformly distributed, then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(s_k) = \int_0^1 f(x)\,dx$$

for every Riemann-integrable $f(x)$.

In this note, I shall mean by a *well-distributed* sequence $\{s_k\}$ one for which

$$\lim_{p\to\infty} \frac{1}{p} \sum_{k=n}^{n+p} I(s_k) = b-a$$

holds uniformly in n. In other words, $\{s_k\}$ is well-distributed if $\{I(s_k)\}$ is almost convergent to $b-a$ for every I. Since any almost convergent sequence is summable $(C, 1)$, a well-distributed sequence is necessarily uniformly distributed. However, the converse is not true.

Let us suppose that a sequence $\{s_n\}$ is uniformly distributed on the interval $(0, 1)$. From this sequence we construct a new sequence by letting $s'_n = 1$ when n is of the form $\nu^3+1,\ldots, \nu^3+\nu$ for all ν and $s'_n = s_n$ otherwise. This new sequence $\{s'_n\}$ is also uniformly distributed, for, if an interval excludes or includes the point 1, the value of $\sum_{k=1}^{n} I(s_k)$ can be changed only by $o(n)$. On the other hand, for any interval $(a, 1)$

$$\frac{1}{\nu} \sum_{\nu^3}^{\nu^3+\nu} I(s_k) = 1,$$

and $\{s_n'\}$ is not well-distributed. Hence, we see that not all uniformly distributed sequences are well-distributed.

THEOREM 2. *If $\{s_k\}$ is well-distributed, then*

$$\lim_{p\to\infty}\frac{1}{p}\sum_{k=n}^{n+p} f(s_k) = \int_0^1 f(x)\,dx$$

holds uniformly in n for every Riemann-integrable $f(x)$.

Proof. The proof of this theorem closely follows that of the corresponding theorem for uniformly distributed sequences. It is true for any finite step-function; two such step functions are selected so that $f_1 \leqslant f \leqslant f_2$ and

$$0 \leqslant \int f_2\,dx - \int f_1\,dx < \epsilon.$$

If we form any matrix $A = (a_{mk})$ by extracting for our nth row one of the transforms in the nth row of our array for 'almost convergence', then

$$\liminf_{m\to\infty}\sum_{k=0}^{\infty} a_{mk} f(s_k) \geqslant \lim_{m\to\infty}\sum_{k=0}^{\infty} a_{mk} f_1(s_k) = \int f_1(x)\,dx = \int f(x)\,dx - \epsilon.$$

$$\limsup_{m\to\infty}\sum_{k=0}^{\infty} a_{mk} f(s_k) \leqslant \int f(x)\,dx + \epsilon.$$

Since ϵ is arbitrary, we have

$$\lim_{m\to\infty}\sum_{k=0}^{\infty} a_{mk} f(s_k) = \int f(x)\,dx.$$

Since this is true for any of the matrices belonging to 'almost convergence', by Theorem 1, $\{f(s_k)\}$ must be almost convergent to $\int f(x)\,dx$.

It is now a corollary of a theorem of Lorentz (**1**) that, if $\{s_k\}$ is well-distributed, then

$$\lim_{m\to\infty}\sum_{k=0}^{\infty} a_{mk} f(s_k) = \int f(x)\,dx$$

for every Riemann-integrable f and for every regular matrix for which

$$\lim_{m\to\infty}\sum_{k=0}^{\infty} |a_{mk} - a_{m,k+1}| = 0.$$

3. I proceed to develop my criterion for well-distributed sequences. If

$$f(x) = e^{2k\pi ix} = e(kx), \text{ say},$$

where k is a positive integer, then $\int f(x)\,dx = 0$. It follows that $\{e(ks_n)\}$ is almost convergent to zero if $\{s_n\}$ is well-distributed, i.e. $\{T(s_n)\}$ is almost convergent to zero, where T is any trigonometric polynomial without constant term.

THEOREM 3. *The sequence $\{s_n\}$ is well-distributed if $\{e(ks_n)\}$ is almost convergent to zero for every positive integral k.*

Proof. Since 'almost convergence' is linear, we see that, if

$$\tau(x) = \tfrac{1}{2}a_0 + \sum_{k=1}^{N} \{a_k e(kx) + b_k e(-kx)\}, \tag{1}$$

then $\{\tau(s_n)\}$ is almost convergent to $\frac{1}{2}a_0$. Now, if $I(x)$ is the characteristic function of any given interval (a, b), we can find functions $\tau_1(x)$, $\tau_2(x)$, both of the form (1), such that

$$\tau_1(x) > I(x) > \tau_2(x), \qquad \int_0^1 [\tau_1(x) - \tau_2(x)]\, dx < \epsilon.$$

We can thus show by an argument similar to the proof of Theorem 2 that $\{I(s_n)\}$ is almost convergent to $b-a$. The proof can now be carried out in the same way as the proof of Theorem 2.

If $s_n = n\xi - [n\xi] = \{n\xi\}$, where ξ is irrational, then it is well known that $\{n\xi\}$ is uniformly distributed. I shall show that $\{n\xi\}$ is well-distributed. Indeed

$$\sum_{\mu=n}^{n+p} e(ks_\mu) = \sum_{\mu=n}^{n+p} e^{2k\mu\pi\xi i} = e^{2kn\pi\xi i}\frac{1-e^{2k(p+1)\pi\xi i}}{1-e^{2k\pi\xi i}}$$

$$= O(1), \quad \text{for } k = 1, 2, 3,\ldots \text{ and for all } n \text{ and } p.$$

Hence $\{n\xi\}$ is well-distributed.

I am grateful to the referee for his comment concerning the definitions.

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ON MEANS OF ENTIRE FUNCTIONS

By Q. I. RAHMAN (*Aligarh*)

[Received 21 February 1956]

1. LET $f(z)$ be an entire function of order ρ and lower order λ. Then, δ and κ being any positive numbers, let

$$\mu_\delta(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\delta\, d\theta,$$

$$m_{\delta,\kappa}(r) = \frac{1}{\pi r^{\kappa+1}}\int_0^r\int_0^{2\pi} |f(xe^{i\theta})|^\delta x^\kappa\, dxd\theta.$$

Let

$$L_{\delta,\kappa} = \limsup_{r\to\infty}\left\{\frac{\mu_\delta(r)}{m_{\delta,\kappa}(r)}\right\}^{1/\log r},$$

$$l_{\delta,\kappa} = \liminf_{r\to\infty}\left\{\frac{\mu_\delta(r)}{m_{\delta,\kappa}(r)}\right\}^{1/\log r}.$$

It is known [(**2**), problem 66; (**3**), 96–99] that for every entire function $L_{2,1} = e^\rho$ and $l_{2,1} = e^\lambda$. I prove here the theorem:

THEOREM. *For every entire function* $L_{\delta,\kappa} = e^\rho$, $l_{\delta,\kappa} = e^\lambda$.

2. Lemmas

LEMMA 1. *If $f(z)$ is regular in $|z| \leqslant R$, and if*

$$z = re^{i\theta}, \qquad 0 \leqslant r < R, \qquad \delta > 0,$$

then

$$|f(z)|^\delta \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)|f(Re^{i\phi})|^\delta}{R^2-2Rr\cos(\theta-\phi)+r^2}\, d\theta. \tag{1}$$

Proof. Case (i). If $f(z)$ has no zeros in $|z| \leqslant R$, then any branch of $\{f(z)\}^\delta$ is regular in $|z| \leqslant R$, and we have, by Poisson's formula,

$$\{f(z)\}^\delta = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)\{f(Re^{i\phi})\}^\delta}{R^2-2Rr\cos(\theta-\phi)+r^2}\, d\theta,$$

from which (1) follows.

Case (ii). Let $f(z)$ have zeros $\alpha_1, \alpha_2,..., \alpha_p$ in $|z| < R$ but no zero on $|z| = R$. Then, using the function which represents $|z| \leqslant R$ on itself and carries $z = 0$ to α_i ($i = 1, 2,..., p$), we can construct a function $\phi(z)$ having the same zeros as $f(z)$ in $|z| < R$ such that $|\phi(z)| < 1$ in $|z| < R$

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and $|\phi(z)| = 1$ on $|z| = R$. Applying Case (i) to $\{f(z)/\phi(z)\}^\delta$ we get the result in this case.

Case (iii). If $f(z)$ has a zero on $|z| = R$, we can establish the result for $|z| = R+\epsilon$ and let $\epsilon \to 0$.

So the result is true in all cases.

LEMMA 2. *Let*

$$A = \limsup_{r\to\infty} \frac{\log\log \mu_\delta(r)}{\log r}, \qquad B = \liminf_{r\to\infty} \frac{\log\log \mu_\delta(r)}{\log r},$$

$$\alpha = \limsup_{r\to\infty} \frac{\log\log m_{\delta,\kappa}(r)}{\log r}, \qquad \beta = \liminf_{r\to\infty} \frac{\log\log m_{\delta,\kappa}(r)}{\log r}.$$

Then $$A = \alpha = \rho, \qquad B = \beta = \lambda.$$

Proof. If $M(r)$ denotes the maximum modulus of $f(z)$ for $|z| = r$, then

$$\mu_\delta(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta \leqslant \{M(r)\}^\delta. \tag{2}$$

From (1) we have

$$|f(re^{i\theta})|^\delta \leqslant \frac{R+r}{R-r}\mu_\delta(R).$$

Taking $r = \frac{1}{2}R$, we get

$$\mu_\delta(R) \geqslant \tfrac{1}{3}\{M(\tfrac{1}{2}R)\}^\delta. \tag{3}$$

From (2) and (3) it follows that $A = \rho$ and $B = \lambda$.

Since $\mu_\delta(x)$ is (1) an increasing function of x,

$$m_{\delta,\kappa}(r) = \frac{2}{r^{\kappa+1}}\int_0^r \mu_\delta(x)x^\kappa \, dx$$

$$\leqslant \frac{2\mu_\delta(r)}{r^{\kappa+1}}\int_0^r x^\kappa \, dx = \frac{2\mu_\delta(r)}{\kappa+1},$$

and we get $$\alpha \leqslant A, \qquad \beta \leqslant B. \tag{4}$$

Further

$$m_{\delta,\kappa}(2r) = \frac{2}{(2r)^{\kappa+1}}\int_0^{2r} \mu_\delta(x)x^\kappa \, dx$$

$$\geqslant \frac{2}{(2r)^{\kappa+1}}\int_r^{2r} \mu_\delta(x)x^\kappa \, dx$$

$$\geqslant \frac{2\mu_\delta(r)}{(2r)^{\kappa+1}}\frac{(2r)^{\kappa+1}-r^{\kappa+1}}{\kappa+1},$$

and we get
$$\alpha \geqslant A, \qquad \beta \geqslant B. \tag{5}$$
Thus, from (4) and (5), $\alpha = A$ and $\beta = B$.

LEMMA 3. *If δ and κ are any positive numbers, $r^{\kappa+1}\mu_\delta(r)$ is a convex function of $r^{\kappa+1}m_{\delta,\kappa}(r)$.*

Proof. We have
$$\begin{aligned}\frac{d(r^{\kappa+1}\mu_\delta(r))}{d(r^{\kappa+1}m_{\delta,\kappa}(r))} &= \frac{(d/dr)(r^{\kappa+1}\mu_\delta(r))}{(d/dr)(r^{\kappa+1}m_{\delta,\kappa}(r))}\\ &= \frac{(\kappa+1)r^\kappa\mu_\delta(r)+r^{\kappa+1}\mu_\delta'(r)}{(d/dr)\Big((1/\pi)\int\limits_0^r\int\limits_0^{2\pi}|f(xe^{i\theta})|^\delta x^\kappa\,dxd\theta\Big)}\\ &= \frac{(\kappa+1)r^\kappa\mu_\delta(r)+r^{\kappa+1}\mu_\delta'(r)}{2r^\kappa(1/2\pi)\int\limits_0^{2\pi}|f(xe^{i\theta})|^\delta\,d\theta}\\ &= \frac{(\kappa+1)r^\kappa\mu_\delta(r)+r^{\kappa+1}\mu_\delta'(r)}{2r^\kappa\mu_\delta(r)}\\ &= \frac{1}{2}\Big\{(\kappa+1)+r\frac{\mu_\delta'(r)}{\mu_\delta(r)}\Big\},\end{aligned}$$
which increases with r since, by (1), $\log\mu_\delta(r)$ is a convex function of $\log r$, and the lemma is proved.

3. Proof of the theorem

If $L_{\delta,\kappa} < \infty$ then, for a positive ϵ and a suitable constant a,
$$\begin{aligned}\log\{r^{\kappa+1}m_{\delta,\kappa}(r)\} &= \int\limits_0^r \frac{x^\kappa\mu_\delta(x)}{x^{\kappa+1}m_{\delta,\kappa}(x)}\,dx\\ &= O(1)+\int\limits_a^r \frac{\mu_\delta(x)}{m_{\delta,\kappa}(x)}\frac{dx}{x}\\ &< O(1)+\int\limits_a^r (L_{\delta,\kappa}+\epsilon)^{\log x}\frac{dx}{x}\\ &< O(1)+\frac{(L_{\delta,\kappa}+\epsilon)^{\log r}}{\log(L_{\delta,\kappa}+\epsilon)}.\end{aligned} \tag{6}$$

From the fact that $\log\mu_\delta(r)$ is a convex function of $\log r$ it follows very easily that
$$\lim_{r\to\infty}\frac{\log m_{\delta,\kappa}(r)}{\log r} = \infty.$$

Hence, from (6), we have $$L_{\delta,\kappa} \geqslant e^{\rho},$$
which obviously holds when $L_{\delta,\kappa} = \infty$.

It follows from Lemma 3 that $\mu_\delta(x)/m_{\delta,\kappa}(x)$ is an increasing function of x and therefore, for $0 < L_{\delta,\kappa} < \infty$,

$$\begin{aligned} \log\{(2r)^{\kappa+1}m_{\delta,\kappa}(2r)\} &> \int_r^{2r} \frac{x^{\kappa}\mu_\delta(x)}{x^{\kappa+1}m_{\delta,\kappa}(x)}\,dx \\ &> \frac{\mu_\delta(r)}{m_{\delta,\kappa}(r)} \int_r^{2r} \frac{dx}{x} \\ &= \frac{\mu_\delta(r)}{m_{\delta,\kappa}(r)} \log 2 \\ &> (L_{\delta,\kappa}-\epsilon)^{\log r}\log 2, \end{aligned}$$

for a sequence of values of r tending to infinity. Consequently

$$L_{\delta,\kappa} \leqslant e^{\rho},$$

which holds when $L_{\delta,\kappa} = 0$. If $L_{\delta,\kappa} = \infty$, the above argument gives $e^{\rho} = \infty$. This proves that $L_{\delta,\kappa} = e^{\rho}$.

The proof that $l_{\delta,\kappa} = e^{\lambda}$ is similar and is omitted.

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THE MINIMUM MODULUS OF FUNCTIONS REGULAR AND OF FINITE ORDER IN THE UNIT CIRCLE

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1. I SHALL examine the association which exists between the minimum and maximum moduli of certain functions regular in the unit circle. The general results which I obtain here are most significant when applied to a function $g(z)$ of finite order

$$\beta = \limsup_{r\to 1} \frac{\log\log M(r,g)}{-\log(1-r)},$$

where $M(r,g) = \max_{|z|=r} |g(z)|$, the minimum modulus function

$$\mu(r,g) = \min_{|z|=r} |g(z)|$$

being compared with $M(r,g)$ for a sequence of values r which increase to 1.

The corresponding problem in the theory of integral functions has been considered in various ways. Littlewood (**4**) proved that, if $G(z)$ is a non-constant integral function of finite order ρ, then

$$\limsup_{r\to\infty} \frac{\log \mu(r, G)}{\log M(r, G)} > -C(\rho), \tag{1.1}$$

where $C(\rho)$ depends only on ρ. For large values of ρ, Hayman (**2**) has shown that

$$C(\rho) < 2{\cdot}19 \log \rho,$$

and that, if $G(z)$ has infinite order, (1.1) must be superseded in general by

$$\limsup_{r\to\infty} \frac{\log \mu(r, G)}{\log M(r, G) \log\log\log M(r, G)} > -2{\cdot}19.$$

The method used by Hayman to prove these results is extended here to apply to functions regular for $|z| < 1$. The modifications which I make in the theory take the form of a number of preliminary lemmas which generalize this earlier work. The main theorem which I prove is

THEOREM 1. *Let $g(z)$ be a function regular and of finite order β for $|z| < 1$. Then, if $\beta > 1$, there is a corresponding constant $K(\beta)$ such that*

$$-K(\beta) \leqslant \limsup_{r\to 1} \frac{\log \mu(r,g)}{\log M(r,g) \log\log M(r,g)} \leqslant 0. \tag{1.2}$$

The symbols K, A, B, etc., which arise in the sequel represent finite positive constants, not always taking the same value at each occurrence and sometimes dependent on certain parameters which will usually be indicated.

Theorem 1 differs from the result quoted above for integral functions in the occurrence of the term $\log\log M(r,g)$ in (1.2). The necessity to include such a term in a general result of this type is shown later when, for each finite value β greater than 1, we construct a function $g(z)$ regular and of order β in $|z| < 1$ for which the limit occurring in (1.2) is non-zero. I remark also that the inequalities (1.2) are not true in general for functions which are regular in $|z| < 1$ but of order not exceeding the value 1. This type of function is not dealt with in any great detail here except to consider briefly how the behaviour in particular cases differs from that of general functions of greater finite order.

2. The earlier lemmas of this paper are proved with respect to a function $f(z)$ regular for $|z| \leqslant 1$ and an associated function $F(w)$, extensions being made which apply to functions $g(z)$ regular for $|z| < 1$. With Hayman I write the function $u(z) = \log|f(z)|$ as

$$u(z) = u(z,h) - N(z,h,f), \tag{2.1}$$

where
$$N(z,h,f) = \int_0^h n(z,t,f)\, t^{-1}\, dt, \tag{2.2}$$

and $n(z,t,f)$ is the number of zeros of the function $f(\xi)$ in the circle $|z-\xi| \leqslant t$. Thus $u(z)$ is expressed as the difference of two components by considering those zeros of $f(\xi)$ which are situated in a certain neighbourhood of the point z. I shall examine each of these terms separately, employing the following two lemmas proved by Hayman [(**2**) 476, 483].

LEMMA A. *Suppose that $F(w)$ is regular for $|w| \leqslant 1$, $F(0) = 1$ and $0 < \rho < 1$, $h = \delta(1-\rho)$, where $0 < \delta < \frac{1}{2}$. Then, if $U(w) = \log|F(w)|$, we have*

$$|U(\rho,h)| \leqslant \frac{A(\delta)}{1-\rho} \int_0^{2\pi} U^+(e^{i\psi})\, d\psi, \tag{2.3}$$

$$n(\rho,h,F) \leqslant \frac{1+2\delta}{1-\rho} \frac{1}{2\pi} \int_0^{2\pi} U^+(e^{i\psi})\, d\psi, \tag{2.4}$$

where $U^+ = \max(U,0)$.

LEMMA B. *Suppose that $0 < r < 1$, $h > 0$ and that for $|z| = r$ we have $n(z,h) \leqslant n_0$. Then there is a set of values σ for which $r < \sigma < r+\frac{1}{2}h$ having measure at least $\frac{1}{4}h$ and such that*

$$N(z, \tfrac{1}{2}h) \leqslant n_0 \log\frac{B}{h},$$

for $|z| = \sigma$.

In Theorem 2 we deduce inequalities for the functions $u(z,h)$ and $n(z,h,f)$. Thence, if $g(z)$ is regular for $|z| < 1$, we obtain a corresponding bound to $\mu(r,g)$ in Theorem 3 in terms of an integral which involves $M(r,g)$. The result is brought into the particular form (1.2) by introducing the concept of 'proximate orders', which is familiar in the general theory of integral functions.

3. Functions regular for $|z| \leqslant 1$

By taking the integral of $U^+(w)$ over the whole of the circle $|w| = 1$ in Lemma A we may, especially if $U^+(w)$ is large on an appreciable portion of that circle, fail to obtain bounds to $|U(\rho,h)|$ and $n(\rho,h,F)$ which lead to results of the best possible order. Roughly speaking it is only that contribution to the integral arising from an appropriate neighbourhood of $\psi = 0$ which is necessary to fix the bounds to $|U(\rho,h)|$ and $n(\rho,h,F)$. Consequently we make a transformation which effectively reduces the range of integration of the integral which occurs in (2.3) and (2.4), thus sharpening the applications of these inequalities. Certain properties of this transformation are given by Lemmas 1, 2, 3.

LEMMA 1. *The transformation*

$$1-z = (1-w)^\alpha, \qquad \tfrac{1}{2} \leqslant \alpha < 1, \tag{3.1}$$

maps the circle $|w| \leqslant 1$ onto a region G which, except for the point $z = 1$, is entirely contained in $|z| < 1$.

Hayman has previously considered the particular mapping given by (3.1) with $\alpha = \frac{1}{2}$. We examine that branch of the right-hand side of (3.1) which is positive when $1-w$ is positive. Thus G is simply-connected and each of its boundary points corresponds to some point $w = e^{i\psi}$ $(0 \leqslant \psi < 2\pi)$ of the circle $|w| = 1$. For such a point z we have by (3.1)

$$z = 1-(1-e^{i\psi})^\alpha = 1-e^{\frac{1}{2}i\alpha(\psi-\pi)}(2\sin\tfrac{1}{2}\psi)^\alpha:$$

that is
$$|z|^2 = 1-2\cos\tfrac{1}{2}\alpha(\psi-\pi)(2\sin\tfrac{1}{2}\psi)^\alpha+(2\sin\tfrac{1}{2}\psi)^{2\alpha}. \tag{3.2}$$

In order to find an upper bound to $|z|$ we examine

$$I(\psi) = 2\cos\tfrac{1}{2}\alpha(\psi-\pi)-(2\sin\tfrac{1}{2}\psi)^\alpha. \tag{3.3}$$

The function $I(\psi)$ is continuous in the interval $(0, 2\pi)$, taking at each of its end points the value

$$I(0) = I(2\pi) = 2\cos\tfrac{1}{2}\pi\alpha = 2\sin\tfrac{1}{2}(1-\alpha)\pi \geqslant 2(1-\alpha), \tag{3.4}$$

since $\frac{1}{2} \leqslant \alpha < 1$. At the turning points of $I(\psi)$ we have

$$\frac{dI}{d\psi} = -\alpha\sin\tfrac{1}{2}\alpha(\psi-\pi)-\alpha\cos\tfrac{1}{2}\psi(2\sin\tfrac{1}{2}\psi)^{\alpha-1} = 0,$$

so that for these values of ψ in the interval $(0, 2\pi)$,

$$\begin{aligned} I(\psi) &= 2\cos\tfrac{1}{2}\alpha(\psi-\pi)+\frac{2\sin\frac{1}{2}\psi\sin\frac{1}{2}\alpha(\psi-\pi)}{\cos\frac{1}{2}\psi} \\ &= 2\frac{\cos\frac{1}{2}\{\psi-(\psi-\pi)\alpha\}}{\cos\frac{1}{2}\psi} = 2\frac{\sin\frac{1}{2}(1-\alpha)(\pi-\psi)}{\sin\frac{1}{2}(\pi-\psi)} \geqslant 2(1-\alpha). \end{aligned}$$

Thus from (3.1), (3.2), (3.3), (3.4) we deduce

$$|z|^2 \leqslant 1-2(1-\alpha)(2\sin\tfrac{1}{2}\psi)^\alpha \leqslant 1,$$

for $w = e^{i\psi}$ $(0 \leqslant \psi < 2\pi)$ with equality holding only if $\psi = 0$. This completes the proof of Lemma 1.

4. We now consider a function $f(z)$ regular for $|z| \leqslant 1$, associating with it a function $F(w)$ to which we can apply Lemma A. Lemmas 2 and 3 contain various relations which exist between the particular functions $f(z)$ and $F(w)$.

LEMMA 2. *Suppose that $f(z)$ is regular for $|z| \leqslant 1$ and that*

$$F(w) = f\{z(w)\} = f\{1-(1-w)^\alpha\}, \qquad \tfrac{1}{2} \leqslant \alpha < 1, \tag{4.1}$$

$$u(z) = \log|f(z)|, \qquad U(w) = \log|F(w)|.$$

Then to each positive value ϵ there corresponds a value $r_0 = r_0(\alpha, \epsilon) < 1$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} U^+(e^{i\psi})\,d\psi \leqslant \frac{1+\epsilon}{\alpha\pi}(\cos\tfrac{1}{2}\pi\alpha)^{-1/\alpha}\int_0^1 M(t, u^+)(1-t)^{1/\alpha-1}\,dt + M(r_0, u^+). \tag{4.2}$$

Each boundary point $w = e^{i\psi}$ of the circle $|w| = 1$ corresponds by (3.1) to a point z where

$$z = re^{i\theta} = a(\psi) = 1-(1-e^{i\psi})^\alpha \quad (0 \leqslant \psi < 2\pi), \tag{4.3}$$

and we can write

$$\int_0^{2\pi} U^+(e^{i\psi})\,d\psi = \int_0^{\pi} u^+[a(\psi)]+u^+[a(2\pi-\psi)]\,d\psi. \tag{4.4}$$

Considering this latter integral we note that, when $0 < \eta \leqslant \psi \leqslant \pi$,

$$|a(\psi)| = |a(2\pi-\psi)| \leqslant r_0 = r_0(\alpha, \eta) < 1,$$

from which it follows that

$$u^+[a(\psi)]+u^+[a(2\pi-\psi)] \leqslant 2M(r_0, u^+).$$

Hence

$$\frac{1}{2\pi}\int\limits_{\eta}^{\pi} u^+[a(\psi)]+u^+[a(2\pi-\psi)]\,d\psi \leqslant \frac{M(r_0, u^+)}{\pi}\int\limits_{\eta}^{\pi} d\psi \leqslant M(r_0, u^+), \tag{4.5}$$

and it remains to consider the contribution to the integral (4.4) which arises from the interval $(0, \eta)$.

By (4.3) and a rearrangement of (3.2) we have

$$1-r^2 = (2\sin\tfrac{1}{2}\psi)^\alpha\{2\cos\tfrac{1}{2}\alpha(\psi-\pi)-(2\sin\tfrac{1}{2}\psi)^\alpha\},$$

and by differentiating we have, as $\psi \to 0$,

$$\frac{d\psi}{dr} \sim -(\alpha\cos\tfrac{1}{2}\pi\alpha)^{-1}(2\sin\tfrac{1}{2}\psi)^{1-\alpha}.$$

Therefore, if $\eta = \eta(\alpha, \epsilon)$ is a sufficiently small positive value,

$$\left|\frac{d\psi}{dr}\right| \leqslant \frac{1+\epsilon}{\alpha}(\cos\tfrac{1}{2}\pi\alpha)^{-1/\alpha}(1-r)^{1/\alpha-1},$$

for $0 \leqslant \psi \leqslant \eta$. Since

$$u^+[a(\psi)]+u^+[a(2\pi-\psi)] \leqslant 2M(r, u^+),$$

it follows that

$$\frac{1}{2\pi}\int\limits_{0}^{\eta} u^+[a(\psi)]+u^+[a(2\pi-\psi)]\,d\psi$$

$$\leqslant \frac{1+\epsilon}{\alpha\pi}(\cos\tfrac{1}{2}\pi\alpha)^{-1/\alpha}\int\limits_{|a(\eta)|}^{1} M(r, u^+)(1-r)^{1/\alpha-1}\,dr$$

$$\leqslant \frac{1+\epsilon}{\alpha\pi}(\cos\tfrac{1}{2}\pi\alpha)^{-1/\alpha}\int\limits_{0}^{1} M(t, u^+)(1-t)^{1/\alpha-1}\,dt.$$

Together with (4.4) and (4.5) this latter gives (4.2).

5. Although the function $F(w)$ is not regular for $|w| \leqslant 1$, its only singularity in that region is at the point $w = 1$, where it is bounded. Hence we can apply Lemmas A and 2 to obtain an upper bound to $n(\rho, h, F)$ and $|U(\rho, h)|$ in terms of $M(t, u^+)$. Corresponding bounds to $n(r, h, f)$ and $u(r, h)$ can be deduced by application of the following lemma.

LEMMA 3. *Let the functions $f(z)$ and $F(w)$ be defined as in Lemma 2 and suppose that* $0 < r < 1$,

$$1-\rho = (1-r)^{1/\alpha}, \tag{5.1}$$

$$0 < h < \tfrac{1}{4}(1-r), \qquad h_1 = \tfrac{1}{2}h(1-r)^{1/\alpha-1}, \qquad h_2 = 3h(1-r)^{1/\alpha-1}.$$

Then
$$n(\rho, h_1, F) \leqslant n(r,h,f) \leqslant n(\rho, h_2, F), \tag{5.2}$$

$$U(\rho, h_1) \leqslant u(r,h) \leqslant U(\rho, h_2). \tag{5.3}$$

The inequalities (5.2) are a consequence of the next lemma which will be proved later in this section. I use the notation of Lemma 3 in the statement.

LEMMA 4. *The transformation* (3.1) *maps each circle*

$$|z-r| \leqslant h \leqslant \tfrac{1}{4}(1-r)$$

onto a region $G(r,h)$ of the w-plane which is contained in the circle

$$|w-\rho| \leqslant h_2$$

and contains $|w-\rho| \leqslant h_1$.

We can now prove Lemma 3 in the same way as Hayman proved a corresponding result [(2) 480, Lemma 3 (ii)]. The number $n(r,h,f)$ of zeros of $f(z)$ in $|z-r| \leqslant h$ is equal to the number of zeros of $F(w)$ in $G(r,h)$. Since the region $G(r,h)$ is contained in $|w-\rho| \leqslant h_2$, by Lemma 4 we have

$$n(r,h,f) \leqslant n(\rho, h_2, f).$$

The remaining inequality of (5.2) is true similarly.

Now by (2.2) we have

$$N(z,h) = \int_0^h n(z,t)\, t^{-1}\, dt = \int_0^1 n(z, \mu h)\, \mu^{-1}\, d\mu,$$

and, by (5.2),

$$n(\rho, \mu h_1, F) \leqslant n(r, \mu h, f) \leqslant n(\rho, \mu h_2, F),$$

for $0 \leqslant \mu \leqslant 1$, $h < \tfrac{1}{4}(1-r)$. Dividing by μ and integrating with respect to this parameter over the interval (0, 1) we obtain

$$N(\rho, h_1, F) \leqslant N(r,h,f) \leqslant N(\rho, h_2, F).$$

The inequalities (5.3) now follow from (2.1) since $u(r) = U(\rho)$ by definition.

In order to complete the proof of Lemma 3 we need to verify Lemma 4. By (3.1) and (5.1) we have

$$w-\rho = (1-r)^{1/\alpha} - (1-z)^{1/\alpha}. \tag{5.4}$$

When $|z-r| = h \leqslant \frac{1}{4}(1-r)$, we have

$$|1-z| \leqslant 1-r+|r-z| \leqslant \tfrac{5}{4}(1-r),$$

$$\left|\frac{dw}{dz}\right| = \frac{1}{\alpha}|1-z|^{1/\alpha-1} < 3(1-r)^{1/\alpha-1},$$

since $\frac{1}{2} \leqslant \alpha < 1$. Hence (5.4) gives

$$|w-\rho| \leqslant |z-r| \max_{|z-r|\leqslant\frac{1}{4}(1-r)} \left|\frac{dw}{dz}\right| < 3h(1-r)^{1/\alpha-1} = h_2,$$

which shows that $G(r,h)$ is contained in $|w-\rho| \leqslant h_2$.

To show that $G(r,h)$ contains $|w-\rho| \leqslant h_1$, we prove that the inverse transformation

$$z = 1-(1-w)^{\alpha}$$

maps the circle $|w-\rho| \leqslant h_1$ onto a region included in $|z-r| \leqslant h$. By (3.1) and (5.1) we have

$$z-r = (1-\rho)^{\alpha}-(1-w)^{\alpha}.$$

Now
$$h_1 = \tfrac{1}{2}h(1-r)^{1/\alpha-1} < \tfrac{1}{8}(1-\rho),$$

so that, when $|w-\rho| = h_1$, we have

$$|1-w| \geqslant 1-\rho-|\rho-w| > \tfrac{7}{8}(1-\rho),$$

$$\left|\frac{dz}{dw}\right| = \alpha|1-w|^{\alpha-1} < \tfrac{8}{7}(1-\rho)^{\alpha-1}.$$

Hence, for these values of w,

$$|z-r| \leqslant |w-\rho| \max_{|w-\rho|<\frac{1}{8}(1-\rho)} \left|\frac{dz}{dw}\right| < \tfrac{8}{7}h_1(1-\rho)^{\alpha-1} = \tfrac{4}{7}h < h,$$

and Lemma 4 is proved.

6. Lemmas 2 and 3 provide with Lemma A the means of proving the main result concerning functions regular for $|z| \leqslant 1$. It is stated thus:

THEOREM 2. *Let $f(z)$ be a function regular for $|z| \leqslant 1$, $f(0) = 1$, and suppose that $\frac{1}{2} \leqslant \alpha < 1$. Then to each positive value ϵ there corresponds a value $r_0 = r_0(\alpha, \epsilon) < 1$ such that for $|z| = r$ and $h = \delta(1-r)$, where $0 < \delta < \frac{1}{6}$, we have*

$$u(z,h) \geqslant -A(\alpha,\delta)(1-r)^{-1/\alpha}\left\{\int_0^1 M(t,u^+)(1-t)^{1/\alpha-1}\,dt + M(r_0,u^+)\right\}, \tag{6.1}$$

$$n(z,h,f) \leqslant (1+6\delta)(1-r)^{-1/\alpha}\times$$
$$\times\left\{\frac{1+\epsilon}{\alpha\pi}(\cos\tfrac{1}{2}\pi\alpha)^{-1/\alpha}\int_0^1 M(t,u^+)(1-t)^{1/\alpha-1}\,dt + M(r_0,u^+)\right\}. \tag{6.2}$$

By applying Lemmas A and 2 to the function $F(w)$ defined by (4.1) we obtain

$$U(\rho, h_1) \geqslant -\frac{A(\alpha, \delta_1)}{1-\rho}\left\{\int_0^1 M(t, u^+)(1-t)^{1/\alpha-1}\,dt + M(r_0, u^+)\right\},$$

for $0 < \rho < 1$, $h_1 = \delta_1(1-\rho)$, $0 < \delta_1 < \frac{1}{2}$. Now, if

$$h = 2h_1(1-r)^{1-1/\alpha} = 2\delta_1(1-\rho)(1-r)^{1-1/\alpha} = 2\delta_1(1-r),$$

the hypotheses of Lemma 3 are satisfied for $\delta = 2\delta_1 < \frac{1}{4}$, and it follows by (5.1) and (5.3) that

$$\begin{aligned} u(r,h) &\geqslant U(\rho, h_1) \\ &\geqslant -A(\alpha,\delta)(1-r)^{-1/\alpha}\left\{\int_0^1 M(t, u^+)(1-t)^{1/\alpha-1}\,dt + M(r_0, u^+)\right\}, \end{aligned}$$

for $h = \delta(1-r)$.

Further from (2.4), (4.2), (5.1), and (5.2) we can deduce that, for

$$h_2 = \delta_2(1-\rho) < \tfrac{1}{2}(1-\rho), \qquad h = \tfrac{1}{3}h_2(1-r)^{1-1/\alpha} = \tfrac{1}{3}\delta_2(1-r),$$

$$\begin{aligned} n(r,h,f) &\leqslant n(\rho, h_2, F) \\ &\leqslant \frac{(1+2\delta_2)}{1-\rho}\left\{\frac{1+\epsilon}{\alpha\pi}(\cos\tfrac{1}{2}\pi\alpha)^{-1/\alpha}\int_0^1 M(t, u^+)(1-t)^{1/\alpha-1}\,dt + M(r_0, u^+)\right\} \\ &\leqslant (1+2\delta_2)(1-r)^{-1/\alpha} \times \\ &\qquad \times \left\{\frac{1+\epsilon}{\alpha\pi}(\cos\tfrac{1}{2}\pi\alpha)^{-1/\alpha}\int_0^1 M(t, u^+)(1-t)^{1/\alpha-1}\,dt + M(r_0, u^+)\right\}, \end{aligned}$$

provided that $\delta_2 < \frac{1}{2}$. If we choose

$$\delta = \tfrac{1}{3}\delta_2 < \tfrac{1}{6},$$

the inequalities (6.1) and (6.2) follow with $z = r$. That the inequalities are true in general can be proved by a similar consideration of the function $f(ze^{i\theta})$.

7. Functions regular for $|z| < 1$

The theory of the preceding sections can be adapted to apply to functions regular for $|z| < 1$ by considering these functions in the circles $|z| \leqslant R < 1$. In this way we extend Theorem 2 and effect the proof of Theorem 1 by making use of certain fundamental results in the theory of proximate orders.

THEOREM 3. *Let $g(z)$ be regular for $|z| < 1, g(0) = 1$, and suppose that $\frac{1}{2} \leqslant \alpha < 1$. Then, if $r \leqslant R < 1$, there is a set of values τ of measure at least $\frac{1}{32}R(1-r)$ in the interval $\{Rr, R(r+\frac{1}{16}(1-r))\}$ and a value*

$$r_0 = r_0(\alpha) < 1$$

such that, when $r_0 \leqslant r < 1$,

$$v(z) = \log|g(z)| \geqslant -(1-r)^{-1/\alpha}\log\frac{A(\alpha)}{1-r}\times$$

$$\times\left\{\frac{8}{\alpha\pi}(R\cos\tfrac{1}{2}\pi\alpha)^{-1/\alpha}\int_0^R M(t,v^+)(R-t)^{1/\alpha-1}\,dt+2M(r_0,v^+)\right\}, \quad (7.1)$$

on each circle $|z| = \tau$.

Since the function
$$f(z) = g(Rz) \quad (7.2)$$
is regular for $|z| \leqslant 1$, we have, by applying Lemma B and Theorem 2 with $h = \delta(1-r)$, $\epsilon = \delta = \frac{1}{8}$, that in the interval $(r, r+\frac{1}{2}h)$ there is a set of values σ of measure at least $\frac{1}{4}h$ for which

$$N(z,\tfrac{1}{2}h,f) \leqslant (1-r)^{-1/\alpha}\log\frac{B}{1-r}\times$$

$$\times\left\{\frac{2}{\alpha\pi}(\cos\tfrac{1}{2}\alpha\pi)^{-1/\alpha}\int_0 M(t,u^+)(1-t)^{1/\alpha-1}\,dt+2M(r_0,u^+)\right\}, \quad (7.3)$$

for $|z| = \sigma$. Hence for such values of z we have, by (2.1) and (6.1),

$$\log|f(z)| = u(z,\tfrac{1}{2}h)-N(z,\tfrac{1}{2}h,f)$$

$$\geqslant -(1-r)^{-1/\alpha}\log\frac{A(\alpha)}{1-r}\times$$

$$\times\left\{\frac{2}{\alpha\pi}(\cos\tfrac{1}{2}\pi\alpha)^{-1/\alpha}\int_0^1 M(t,u^+)(1-t)^{1/\alpha-1}\,dt+2M(r_0,u^+)\right\}.$$

From (7.2) it follows that for $|z| = R\sigma = \tau$ we have

$$\log|g(z)| = \log|f(zR^{-1})|$$

$$\geqslant -(1-r)^{-1/\alpha}\log\frac{A(\alpha)}{1-r}\times$$

$$\times\left\{\frac{2}{\alpha\pi}(\cos\tfrac{1}{2}\pi\alpha)^{-1/\alpha}\int_0^1 M(t,u)(1-t^+)^{1/\alpha-1}\,dt+2M(r_0,u^+)\right\}. \quad (7.4)$$

But, since $u(z) = v(Rz)$ by definition, it follows that

$$M(t, u^+) = M(Rt, v^+).$$

Hence we have

$$\int_0^1 M(t, u^+)(1-t)^{1/\alpha-1}\,dt = \int_0^1 M(Rt, v^+)(1-t)^{1/\alpha-1}\,dt$$

$$= R^{-1/\alpha}\int_0^R M(t, v^+)(R-t)^{1/\alpha-1}\,dt.$$

The inequality (7.1) now follows from (7.4) since $M(r_0, u^+) \leqslant M(r_0, v^+)$.

The set of values σ for which (7.3) holds on $|z| = \sigma$ has measure at least $\frac{1}{4}\delta(1-r)$ in the interval $\{r, r+\frac{1}{2}\delta(1-r)\}$. Because we have chosen $\delta = \frac{1}{8}$, the set of values τ for which (7.1) holds on $|z| = \tau$ has measure at least $\frac{1}{32}R(1-r)$ in the interval $\{Rr, R(r+\frac{1}{16}(1-r))\}$. This completes the proof of Theorem 3.

8. Theorem 3 is a general result which is valid for all functions regular in the circle $|z| < 1$. We now deduce Theorem 1 as a particular application to functions of finite order greater than 1. It is required to associate the integral which occurs in (7.1) more directly with the function $M(r, g)$ and we proceed as follows.

A function $\rho(x)$ is said to be a *Lindelöf proximate order* if it satisfies the conditions [(**1**) 54]

(i) $\rho(x)$ *is real-valued, continuous, and differentiable in adjacent intervals for* $x > x_0$;

(ii) $\lim_{x\to\infty} \rho'(x)x\log x = 0$;

(iii) $\lim_{x\to\infty} \rho(x) = \rho$ $(0 \leqslant \rho < \infty)$.

Proximate orders provide a means of approximating to the maximum modulus of integral functions of finite order, and it has been proved that to each such function $G(z)$ there corresponds a proximate order $\rho(x)$ such that

$$\limsup_{r\to\infty} r^{-\rho(r)}\log m(r, G) = 1.$$

In proving this result, Shah (**5**) has not needed many special properties of the function $M(r, G)$. In fact he has based his proof solely on the continuity of $M(r, G)$ and the fact that for functions of finite order,

$$\limsup_{r\to\infty}\frac{\log\log M(r, G)}{\log r},$$

is finite. Thus his proof covers more than he has actually stated and in particular

THEOREM 4. *Let $\Psi(x)$ be a positive continuous function defined for $x > x_0$, such that*

$$\limsup_{x\to\infty} \frac{\log \Psi(x)}{\log x} = \rho < \infty, \quad \rho \geqslant 0.$$

Then there is a Lindelöf proximate order $\rho(x)$ such that

$$\limsup_{x\to\infty} x^{-\rho(x)}\Psi(x) = 1.$$

As a corollary we have a corresponding theorem for functions regular in the unit circle.

THEOREM 5. *Let $g(z)$ be regular, unbounded, and of finite order for $|z| < 1$. Then there is a Lindelöf proximate order $\rho(x) = \beta(x^{-1})$ such that*

$$\limsup_{r\to 1} (1-r)^{\beta(1-r)} \log M(r,g) = 1.$$

For the function

$$\Psi(x) = \log M(1-x^{-1}, g)$$

satisfies the hypothesis of Theorem 4. Hence there is a Lindelöf proximate order $\rho(x)$ such that

$$\limsup_{x\to\infty} x^{-\rho(x)} \log M(1-x^{-1}, g) = 1.$$

If we write $x = (1-r)^{-1}$, $\beta(x^{-1}) = \rho(x)$, we have

$$\limsup_{r\to 1} (1-r)^{\beta(1-r)} \log M(r,g) = 1,$$

thus verifying Theorem 5.

A further property of proximate orders which will be required later is that, for each positive constant k,

$$(kx)^{\rho(kx)} \sim k^{\rho} x^{\rho(x)}, \tag{8.1}$$

as x tends to infinity [(**1**) 55, Lemma 2]. This can be deduced from (i), (ii), (iii).

9. If $\rho(x)$ is a Lindelöf proximate order associated as in Theorem 5 with the function $g(z)$ of Theorem 1 and

$$\beta(t) = \rho(t^{-1}), \tag{9.1}$$

then it follows from (iii) that the order of $g(z)$ is by hypothesis

$$\limsup_{r\to 1} \beta(1-r) = \lim_{r\to 1} \beta(1-r) = \beta > 1. \tag{9.2}$$

Before making our application of Theorem 3 we proceed to replace the integral which occurs on the right-hand side of (7.1) by a more convenient expression by writing $R = r$.

Theorem 5 shows that to each constant C greater than unity there corresponds a value r_0 such that

$$M(r,v^+) = \log M(r,g) < C(1-r)^{-\beta(1-r)} \quad (r_0 < r < 1). \tag{9.3}$$

If r_0 is chosen appropriately, then by (iii), (9.1), and (9.2) the function [cf. (1) 55, Lemma 1]

$$(1-r)^{\frac{1}{2}(\beta+1)-\beta(1-r)}$$

increases with r for $r_0 \leqslant r < 1$. We now define

$$\alpha^{-1} = \begin{cases} \frac{1}{4}(3+\beta) & (1 < \beta \leqslant 5), \\ 2 & (5 < \beta < \infty), \end{cases}$$

and note that $\frac{1}{2} \leqslant \alpha < 1$. We have also

$$\alpha^{-1} \leqslant \tfrac{1}{4}(3+\beta) \leqslant \tfrac{1}{2}(1+\beta) \quad (1 < \beta < \infty), \tag{9.4}$$

so that, by (9.3),

$$\begin{aligned} \int_{r_0}^{r} M(t,v^+)(r-t)^{1/\alpha-1}\,dt &< C\int_{r_0}^{r} (1-t)^{1/\alpha-1-\beta(1-t)}\,dt \\ &< C(1-r)^{\frac{1}{2}(1+\beta)-\beta(1-r)} \int_{r_0}^{r} (1-t)^{1/\alpha-\frac{1}{2}(3+\beta)}\,dt \\ &< K(\beta)(1-r)^{1/\alpha-\beta(1-r)}, \end{aligned}$$

if, for example, we take $C = 2$. Further

$$\int_{0}^{r_0} M(t,v^+)(r-t)^{1/\alpha-1}\,dt \leqslant M(r_0,v^+)\int_{0}^{r_0} (r-t)^{1/\alpha-1}\,dt < M(r_0,v^+).$$

Thus for a suitable value r_0 we have

$$\int_{0}^{r} M(t,v^+)(r-t)^{1/\alpha-1}\,dt < K(\beta)(1-r)^{1/\alpha-\beta(1-r)} + M(r_0,v^+), \tag{9.5}$$

for $r_0 < r < 1$.

Now, if $R = r$, the interval

$$\{Rr, R(r+\tfrac{1}{16}(1-r))\}$$

of Theorem 3 is contained in $(2r-1,r)$ for sufficiently small values of $1-r$. Hence the application of this theorem gives that in each interval $(2r-1,r)$ there is a value τ for which

$$\log\mu(\tau,g) > -K(\beta)\log\frac{A}{1-r}\{(1-r)^{-\beta(1-r)} + M(r_0,v^+)(1-r)^{-1/\alpha}\}, \tag{9.6}$$

by (9.5). Here we assume without loss of generality that $g(0) = 1$.

By Theorem 5 we have that for each constant D less than unity there is an infinite sequence of values s increasing to 1 such that

$$\log M(s,g) > D(1-s)^{-\beta(1-s)}. \tag{9.7}$$

We may write any typical value s as $2r-1$, and have that (9.6) holds for a value τ in the interval $(s, \frac{1}{2}(1+s))$. Since

$$1-\tau \leqslant 1-s = 2(1-r),$$

we have
$$\log M(\tau,g) \geqslant \log M(s,g),$$

and it follows from (8.1) and (9.7) that, if $1-s$ is sufficiently small.

$$\log M(\tau,g) > D(1-s)^{-\beta(1-s)} > K(\beta)(1-r)^{-\beta(1-r)},$$

$$\operatorname{loglog} M(\tau,g) > K(\beta)\log\frac{1}{1-r},$$

for $D = \frac{1}{2}$, say. Together with (9.6) these latter inequalities give

$$\frac{\log\mu(\tau,g)}{\log M(\tau,g)} > -K(\beta)\left(\log\frac{1}{1-r}\right)\{1+o(1)\}$$

$$> -K(\beta)\operatorname{loglog} M(\tau,g)\{1+o(1)\}. \tag{9.8}$$

Because (9.7) holds for a sequence of values s increasing to 1, (9.8) holds for a corresponding sequence of values τ also increasing to 1, and therefore

$$\limsup_{r\to 1}\frac{\log\mu(r,g)}{\log M(r,g)\operatorname{loglog} M(r,g)} > -K(\beta).$$

Since $g(z)$ is of positive order, the function $\operatorname{loglog} M(r,g)$ increases to infinity as $r \to 1$. Hence, if r is sufficiently close to the value 1,

$$\frac{\log\mu(r,g)}{\log M(r,g)\operatorname{loglog} M(r,g)} < \frac{\log M(r,g)}{\log M(r,g)\operatorname{loglog} M(r,g)} = \frac{1}{\operatorname{loglog} M(r,g)},$$

and we have
$$\limsup_{r\to 1}\frac{\log\mu(r,g)}{\log M(r,g)\operatorname{loglog} M(r,g)} \leqslant 0,$$

thus completing the proof of Theorem 1.

10. A counter-example for Theorem 1

In order to show that in a sense the result of Theorem 1 is best-possible I construct a particular function for which the limit occurring in (1.2) is negative. Thus we cannot in general replace the factor $\operatorname{loglog} M(r,g)$ by any other function which tends to infinity much less rapidly without affecting the finiteness of this limit.

THEOREM 6. *To each finite value β ($\geqslant 1$) there corresponds a function $\pi(z)$ regular and of order β in $|z| < 1$ such that*

$$\limsup_{r\to 1} \frac{\log \mu(r,\pi)}{\log M(r,\pi) \log\log M(r,\pi)} < 0. \tag{10.1}$$

For the proof of Theorem 6, I construct a function regular for $|z| < 1$ with zeros of multiplicity $[2^{\beta n}]$ situated at the points

$$a_{n,k} = (1-2^{-n}+k\,2^{-2n-1})e^{2\pi i k\, 2^{-n}} \quad (1 \leqslant k \leqslant 2^n;\ 1 \leqslant n < \infty). \tag{10.2}$$

This function is defined as

$$\pi_p(z) = \prod_{n=1}^{\infty} \prod_{k=1}^{2^n} \left[\{1-Z(z,a_{n,k})\} \exp \sum_{m=1}^{p} \frac{1}{m} Z(z,a_{n,k})^m\right]^{[2^{\beta n}]},$$

where $p = [\beta+1]$ and

$$Z(z,a) = 1-\frac{|a|(a-z)}{a(1-z\bar{a})}. \tag{10.3}$$

It will be assumed for convenience that $1 \leqslant \beta < 2$ since the proofs of Lemmas 5, 6, 7 need only trivial amendments to apply in the general case.

LEMMA 5. *If $1 \leqslant \beta < 2$, the function*

$$\pi(z) = \pi_2(z) = \prod_{n=1}^{\infty} \prod_{k=1}^{2^n} \{(1-Z(z,a_{n,k}))\exp(Z(z,a_{n,k})+\tfrac{1}{2}Z(z,a_{n,k})^2)\}^{[2^{\beta n}]}$$

is regular for $|z| < 1$.

We must show that this latter product is uniformly convergent in each circle $|z| \leqslant \rho < 1$. Writing $z = re^{i\theta}$ and $a = |a|e^{i\alpha}$ in (10.3) we have

$$Z(z,a) = \frac{(1-|a|)(1+ze^{-i\alpha})}{1-z\bar{a}}. \tag{10.4}$$

When $|z| = r < 1$, $|a| < 1$, (10.4) gives

$$|Z(z,a)| < \frac{2(1-|a|)}{1-r}, \tag{10.5}$$

and, when $$|a| > \tfrac{1}{4}(3+\rho) \geqslant \tfrac{1}{4}(3+r),$$

we have $|Z(z,a)| < \frac{1}{2}$. We define

$$\mathscr{E}(z,a) = \{1-Z(z,a)\}\exp\{Z(z,a)+\tfrac{1}{2}Z(z,a)^2\}.$$

Then, taking the appropriate branch of the logarithm, we have, by a Taylor expansion,

$$|\log \mathscr{E}(z,a)| \leqslant \sum_{m=3}^{\infty} \frac{1}{m}|Z(z,a)|^m < |Z(z,a)|^3 \sum_{m=1}^{\infty} 2^{-m}:$$

that is $$|\log \mathscr{E}(z,a)| \leqslant |Z(z,a)|^3, \tag{10.6}$$

for $|z| = r \leqslant \rho < 1$, $|a| > \frac{1}{4}(3+\rho)$.

We can choose a finite value n_0 such that

$$|a_{n,k}| = r_{n,k} > \tfrac{1}{4}(3+\rho),$$

for $n > n_0$ and all corresponding values of k. Then it follows from (10.2), (10.5), (10.6) that, for $n_2 \geqslant n_1 \geqslant n_0$, we have

$$\sum_{n=n_1}^{n_2} \sum_{k=1}^{2^n} [2^{\beta n}]|\log \mathscr{E}(z, a_{n,k})| \leqslant \sum_{n=n_1}^{n_2} \sum_{k=1}^{2^n} 2^{\beta n+3}\left(\frac{1-r_{n,k}}{1-\rho}\right)^3$$

$$\leqslant \frac{8}{(1-\rho)^3} \sum_{n=n_1}^{n_2} 2^{(\beta-2)n}.$$

For fixed ρ this latter sum converges uniformly as $n_2 \to \infty$, and therefore the product function $\pi(z)$ is regular for $|z| \leqslant \rho < 1$ for each ρ, which proves the lemma.

11. An upper bound to $M(r, \pi)$ is stated in

Lemma 6. *The function $\pi(z)$ satisfies the inequality*

$$\log M(r, \pi) < K(\beta)2^{\beta N},$$

when $\quad 1-2^{-N+1} \leqslant r \leqslant 1-2^{-N} \quad (N = 2, 3,\ldots). \qquad (11.1)$

For fixed θ we write amp $a_{n,k} = \alpha_{n,k}$, where $|\theta-\alpha_{n,k}| \leqslant \pi$, and express $\pi(z)$ as the product of the five factors

$$P_1(z) = \prod_{n=1}^{N+1} \prod_{k=1}^{2^n} \{1-Z(z, a_{n,k})\}^{[2^{\beta n}]},$$

$$P_2(z) = \prod \exp[[2^{\beta n}]\{Z(z, a_{n,k})+\tfrac{1}{2}Z(z, a_{n,k})^2\}]$$
$$(1 \leqslant n \leqslant N+1;\ |\theta-\alpha_{n,k}| \leqslant \pi 2^{-n}),$$

$$P_3(z) = \prod \exp[[2^{\beta n}]\{Z(z, a_{n,k})+\tfrac{1}{2}Z(z, a_{n,k})^2\}]$$
$$(1 \leqslant n \leqslant N+1;\ |\theta-\alpha_{n,k}| > \pi 2^{-n}),$$

$$P_4(z) = \prod \mathscr{E}(Z, a_{n,k})^{[2^{\beta n}]} \quad (n \geqslant N+2;\ |\theta-\alpha_{n,k}| \leqslant \pi 2^{-N}),$$

$$P_5(z) = \prod \mathscr{E}(z, a_{n,k})^{[2^{\beta n}]} \quad (n \geqslant N+2;\ |\theta-\alpha_{n,k}| > \pi 2^{-N}).$$

When $|z| < 1$, $|a| < 1$, we have, by (10.3),

$$|1-Z(z,a)|^2 = \left|\frac{a-z}{1-z\bar{a}}\right|^2 = \frac{r^2-2r|a|\cos(\theta-\alpha)+|a|^2}{1-2r|a|\cos(\theta-\alpha)+r^2|a|^2} < 1.$$

Hence the modulus of each factor of $P_1(z)$ is less than 1, and in particular we have

$$\log|P_1(z)| \leqslant 0, \qquad (11.2)$$

for $|z| \leqslant 1-2^{-N}$.

From (10.4) we have

$$\begin{aligned}\operatorname{re} Z(z,a) &= \operatorname{re}\frac{(1-|a|)(1+ze^{-i\alpha})(1-\bar{z}a)}{(1-z\bar{a})(1-\bar{z}a)}\\ &= \frac{(1-|a|)\{1+r\cos(\theta-\alpha)-r|a|\cos(\theta-\alpha)-r^2|a|\}}{1-2r|a|\cos(\theta-\alpha)+r^2|a|^2}\\ &\leqslant \frac{2(1-|a|)(1-r|a|)}{1-2r|a|\cos(\theta-\alpha)+r^2|a|^2}.\end{aligned}$$

Also $\qquad \operatorname{re} Z(z,a)^2 \leqslant |Z(z,a)|^2 < \dfrac{4(1-|a|)^2}{1-2r|a|\cos(\theta-\alpha)+r^2|a|^2},$

which gives

$$\operatorname{re}\{Z(z,a)+\tfrac{1}{2}Z(z,a)^2\} < \frac{4(1-|a|)(1-r|a|)}{1-2r|a|\cos(\theta-\alpha)+r^2|a|^2}. \tag{11.3}$$

In considering this latter expression we shall need later the inequalities

$$1-2r|a|\cos(\theta-\alpha)+r^2|a|^2 = (1-r|a|)^2+4r|a|\sin^2\tfrac{1}{2}(\theta-\alpha) \geqslant (1-r|a|)^2, \tag{11.4}$$

and, if $|\theta-\alpha| \leqslant \pi$,

$$1-2r|a|\cos(\theta-\alpha)+r^2|a|^2 \geqslant 4r|a|\sin^2\tfrac{1}{2}(\theta-\alpha) \geqslant (2\pi^{-1})^2r|a|(\theta-\alpha)^2. \tag{11.5}$$

For each factor of $P_2(z)$ we have, by (11.3) and (11.4),

$$\operatorname{re}[2^{\beta n}]\{Z(z,a_{n,k})+\tfrac{1}{2}Z(z,a_{n,k})^2\} < 4.2^{\beta n}. \tag{11.6}$$

With each value n ($\leqslant N+1$) there are associated at most two values of k which correspond to distinct factors of $P_2(z)$. Thus from (11.6) we have

$$\log|P_2(z)| \leqslant \sum_{n=1}^{N+1} 8.2^{\beta n} < 8.2^{\beta(N+2)}. \tag{11.7}$$

Next we consider the factors of $P_3(z)$ and in particular those for which the corresponding value $\alpha_{n,k}$ satisfies

$$\pi 2^{-n} k' < |\theta-\alpha_{n,k}| \leqslant \pi 2^{-n}(k'+1), \tag{11.8}$$

where k' in an integer and

$$1 \leqslant k' \leqslant 2^n-1. \tag{11.9}$$

By (11.5) and (11.8) we have

$$1-2rr_{n,k}\cos(\theta-\alpha_{n,k})+r^2r_{n,k}^2 \geqslant \{(\theta-\alpha_{n,k})/\pi\}^2 > (k')^2 2^{-2n},$$

since by hypothesis $r \geqslant \frac{1}{2}$, $r_{n,k} > \frac{1}{2}$. For each value of n and k' there

are at most two values k which satisfy (11.8), and for each such value of k we have, by (11.3),

$$\operatorname{re}[2^{\beta n}]\{Z(z, a_{n,k})+\tfrac{1}{2}Z(z, a_{n,k})^2\} < 8.2^{\beta n}(1-r_{n,k})(1-r\,r_{n,k})2^{2n}(k')^{-2}$$
$$\leqslant 40.2^{\beta n}(k')^{-2},$$

since $\qquad 1-r_{n,k} \leqslant 2^{-n}, \qquad 1-r\,r_{n,k} < 2^{-n}+2^{-N+1} \leqslant 5.2^{-n}.$

Summation with respect to k' over the range (11.9) and with respect to n for $1 \leqslant n \leqslant N+1$, leads to

$$\log|P_3(z)| \leqslant \sum_{n=1}^{N+1} \sum_{k'=1}^{2^n-1} 80.2^{\beta n}(k')^{-2} < K\sum_{n=1}^{N+1} 2^{\beta n} < K(\beta)2^{\beta N}, \tag{11.10}$$

since $\sum_{k'=1}^{\infty}(k')^{-2}$ is convergent.

If $n \geqslant N+2$, then, by (10.2) and (11.1),

$$r_{n,k} \geqslant 1-2^{-N-2} \geqslant 1-\tfrac{1}{4}(1-r) = \tfrac{1}{4}(3+r).$$

Hence (10.6) is valid with $a = a_{n,k}$, and we have

$$\log|\mathscr{E}(z, a_{n,k})| \leqslant \left(\frac{1-r_{n,k}}{1-r}\right)^3 \leqslant 2^{3(N-n)}. \tag{11.11}$$

To each value n ($\geqslant N+2$) there correspond at most $2^{n-N-1}+1$ distinct values $a_{n,k}$ for which

$$|\theta-\alpha_{n,k}| \leqslant \pi\,2^{-N},$$

and consequently at most $2^{n-N-1}+1$ distinct factors $\mathscr{E}(z, a_{n,k})$ of $P_4(z)$ each of multiplicity $[2^{\beta n}]$. Thus, summing for $n \geqslant N+2$, we have

$$\log|P_4(z)| \leqslant \sum_{n=N+2}^{\infty} 2^{\beta n}(2^{n-N-1}+1)2^{3(N-n)}$$
$$< 2^{2N}\sum_{n=N+2}^{\infty} 2^{(\beta-2)n};$$

that is $\qquad \log|P_4(z)| < K(\beta)2^{\beta N}. \qquad (11.12)$

It remains to examine the function $P_5(z)$ and we consider those factors for which

$$\pi k'\,2^{-N} < |\theta-\alpha_{n,k}| \leqslant \pi(k'+1)2^{-N}, \tag{11.13}$$

where k' is an integer such that

$$1 \leqslant k' \leqslant 2^N-1. \tag{11.14}$$

The inequality (10.6) is again valid with $a = a_{n,k}$, so that by (10.4), (11.5), (11.13), since $r \geqslant \frac{1}{2}$, $r_{n,k} > \frac{1}{2}$,

$$\begin{aligned}\log|\mathscr{E}(z, a_{n,k})| &\leqslant |Z(z, a_{n,k})|^3 \\ &\leqslant 8(1-r_{n,k})^3\{1-2rr_{n,k}\cos(\theta-\alpha_{n,k})+r^2r_{n,k}^2\}^{-\frac{3}{2}} \\ &\leqslant 8.2^{-3n}\pi^3(\theta-\alpha_{n,k})^{-3} \\ &< 8.2^{3(N-n)}(k')^{-3}:\end{aligned}$$

that is
$$\log|\mathscr{E}(z, a_{n,k})| < 8.2^{3(N-n)}(k')^{-3}. \tag{11.15}$$

For each value n ($\geqslant N+2$) and k' there are, by (11.13), at most 2^{n-N} corresponding values of k, each of which is associated with a factor of $P_5(z)$ of multiplicity $[2^{\beta n}]$. Hence by summing over the range (11.14) of k' and for $N+2 \leqslant n < \infty$ respectively, we have, by (11.15),

$$\begin{aligned}\log|P_5(z)| &< \sum_{n=N+2}^{\infty}\sum_{k'=1}^{2^N-1} 8.2^{\beta n}.2^{2(N-n)}(k')^{-3} \\ &\leqslant K\sum_{n=N+2}^{\infty} 2^{2N}.2^{(\beta-2)n},\end{aligned}$$

since $\sum_{k=1}^{\infty} k^{-3}$ is bounded. Hence

$$\log|P_5(z)| < K(\beta)2^{\beta N}, \tag{11.16}$$

and, by (11.2), (11.7), (11.10), (11.12), we have

$$\log|\pi(z)| = \log|P_1P_2P_3P_4P_5| < K(\beta)2^{\beta N},$$

which verifies the inequality stated in Lemma 6.

12. As a final preliminary to the proof of Theorem 6 we need an upper bound to the function $\mu(r, \pi)$. This is provided by

LEMMA 7. *The function $\pi(z)$ satisfies the inequality*

$$\log\mu(r, \pi) < -2^{-\beta(N-1)}\log 2^{N-1}+K(\beta)2^{\beta N}, \tag{12.1}$$

when r satisfies (11.1).

We again use the notation of Lemma 6 and indeed several of the inequalities which were proved there. However, we must amend (11.2). On each circle

$$|z| = 1-2^{-N+1}+k\,2^{-2N+1} \quad (k = 1, 2,..., 2^{N-1}-1),$$

there is a zero of $\pi(z)$ of order $[2^{\beta(N-1)}]$. Hence on each circle $|z| = r$, where r satisfies (11.1), there is some point, say z', whose distance from

at least one of these zeros is not greater than 2^{-2N+1}. Denoting such a zero by a' we have

$$\left|\frac{a'-z'}{1-z'\bar{a}'}\right| < \frac{2^{-2N+1}}{1-|z'|} \leqslant 2^{-N+1}.$$

Since the order of the zero is $[2^{\beta(N-1)}]$, the corresponding term of $\log|P_1(z')|$ is less than $[2^{\beta(N-1)}]\log 2^{-N+1}$, and, since all the other terms are negative, we have

$$\log|P_1(z')| < -2^{\beta(N-1)}\log 2^{N-1}+\log 2^{N-1}.$$

The inequalities (11.7), (11.10), (11.12), (11.16) are valid with $z = z'$, and therefore

$$\log|\pi(z')| < -2^{\beta(N-1)}\log 2^{N-1}+K(\beta)\,2^{\beta N},$$

which gives (12.1).

We can now prove Theorem 6 for, by Lemma 6,

$$\log M(r,\pi)\log\log M(r,\pi) < K(\beta)\,2^{\beta N}\log 2^{N}, \tag{12.2}$$

for $1-2^{-N+1} \leqslant r \leqslant 1-2^{-N}$, $N > 1$. Also by (10.2), the number of zeros of $\pi(z)$ in the circle $|z| \leqslant r < 1$ is greater than $(1-r)^{-\beta-1}$ for $r > \frac{3}{4}$. Thus $\pi(z)$ is of unbounded characteristic in the unit circle since β is positive and consequently $M(r,\pi) \to \infty$ as $r \to 1$. Because of this the left-hand side of (12.2) is positive and with (12.1) we have

$$\frac{\log\mu(r,\pi)}{\log M(r,\pi)\log\log M(r,\pi)} < -\frac{2^{-\beta}(\log 2^{N-1}+K(\beta))}{K(\beta)\log 2^{N}},$$

for $1-2^{-N+1} \leqslant r \leqslant 1-2^{-N}$, $N \geqslant N_0$, if N_0 is sufficiently large. Thus as $N \to \infty$ we have that

$$\limsup_{r\to 1}\frac{\log\mu(r,\pi)}{\log M(r,\pi)\log\log M(r,\pi)} \leqslant -K(\beta) < 0,$$

the result stated in Theorem 6.

13. Functions of order unity

Theorem 1 is not in general true if we relax the hypothesis that $g(z)$ should be of order greater than 1. For example we may consider the function

$$g_1(z) = \exp\left(-\frac{1}{1-z}\right)\left(\log_m\frac{k}{1-z}\right)^{\beta},$$

where m is a positive integer, $\beta > -2$,

$$\log_m x = \log(\log_{m-1}x) \qquad \log_1 x = \log x,$$

and k is a positive value large enough to ensure the regularity of $g_1(z)$ for $|z| < 1$. The function $g_1(z)$ is of order unity and it has been shown

[(3) Theorem 2] that

$$\log \mu(r, g_1) = -\frac{1}{1-r}\left(\log_m \frac{k}{1-r}\right)^{\beta},$$

$$\log M(r, g_1) < \frac{K(m, \beta)}{1-r} \prod_{\mu=1}^{m} \left(\log_\mu \frac{1}{1-r}\right)^{-2} \left(\log_m \frac{1}{1-r}\right)^{\beta},$$

for $r_0 < r < 1$. Hence it follows that

$$\frac{\log \mu(r, g_1)}{\log M(r, g_1) \log\log M(r, g_1)} < -K(m, \beta)\left(\log \frac{1}{1-r}\right) \prod_{\mu=2}^{m} \left(\log_\mu \frac{1}{1-r}\right)^{2},$$

and the limit which appears in (1.2) has the value $-\infty$.

The function $g_1(z)$ represents an extreme example of the exceptional behaviour to be encountered amongst certain functions of order unity which possess no zeros in the unit circle. An earlier theorem [(**3**) Theorem 2] can be applied to such functions to give

THEOREM A. *Let $g(z)$ be regular and possess no zeros in the circle $|z| < 1$. Then, if $g(0) = 1$ and*

$$\log M(r, g) < \frac{A}{1-r} \prod_{\mu=1}^{m} \left(\log_\mu \frac{1}{1-r}\right)^{-2} \prod_{\mu=m}^{n} \left(\log_\mu \frac{1}{1-r}\right)^{\alpha_\mu} \quad (r_0 < r < 1),$$

where each value α_μ is real and $\alpha_m > 0$, there is a constant

$$K = K(m, n, r_0, \alpha_m, \ldots, \alpha_n)$$

such that

$$\log \mu(r, g) > -\frac{KA}{1-r} \prod_{\mu=m}^{n} \left(\log_\mu \frac{1}{1-r}\right)^{\alpha_\mu},$$

for $r_0 < r < 1$.

Although Theorem 3 is valid for each function of order unity, the best-possible general minimum-modulus inequality which can be obtained from it is that for each positive value ϵ and an appropriate constant K,

$$\log \mu(r, g) > -K(1-r)^{-1-\epsilon},$$

for a set of values of r which increases to 1. For an improvement of this inequality in particular cases a result similar to Theorem A can be obtained for functions which possess zeros by applying the methods of the earlier sections of this paper if we replace (3.1) by the transformation

$$1-w = K(1-z)\left(\log_m \frac{k}{1-z}\right)^{\beta},$$

where $\beta > 0$, m is a positive integer and k and K are suitable constants. Such amendments do not lead to any particularly interesting general

results similar to Theorem 1, and I do not proceed further with this aspect of the theory here.

The constant $K(\beta)$ which occurs in (1.2) has not been calculated in any detail, but a reference to (7.1) and (9.3) shows that the particular $K(\beta)$ is finite for each finite value β greater than 1, and tends to infinity as $\beta \to 1$. Although this has not been proved for the function $\pi(z)$ constructed for the proof of Theorem 6, our brief examination of functions of order unity suggests that $K(\beta)$ may not be uniformly bounded in general as $\beta \to 1$.

I am indebted to Mr. Hayman who pointed out to me the methods pursued in the earlier sections of the paper.

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A NOTE ON THE APPROXIMATE SOLUTION OF THE EQUATIONS OF POISSON AND LAPLACE BY FINITE DIFFERENCE METHODS

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1. Introduction

ONE of the numerical methods of finding approximate solutions to Poisson's equation, Laplace's equation, and to other partial differential equations in two dimensions is to replace the continuous independent variable by a set of points arranged on a regular grid, usually square, and the continuous dependent variable by a function defined on this set of grid points. The differential equation is thus replaced by a difference equation, which is then solved by relaxation methods.

If little is known about the solution sought, it is customary to take a coarse mesh, solve the finite difference problem for this grid, and then decrease the mesh size by a factor $1/\sqrt{2}$ by introducing grid points at the centres of the squares of the old grid. This produces a mesh of squares which is orientated at 45° to the axes of the old grid, usually parallel to some rectilinear boundaries. It is convenient therefore to refer to the old grid as a 'square' grid, i.e. parallel to and at right angles to the boundaries, and to refer to the new grid as a 'diagonal' grid.

The usual procedure uses the values for the old grid to set up approximate values at the new points, so as to give zero residuals at the new points [Hartree (4)]; but thereby significant residuals are created at the old points. If the finite-difference problem for this 'diagonal' grid is solved, to a given number of significant figures, by relaxation, it will usually be necessary to modify the pivotal values at both the old and the new points. The process may now be repeated by decreasing the mesh size again by a factor $1/\sqrt{2}$, solving the new difference problem on what is now a 'square' grid, and so on until no significant change takes place in the pivotal values between one stage and the next.

In following this method numerically for a number of problems, we have noticed two significant facts First, for the finer meshes, several extra figures have to be carried to allow for rounding errors, which may be as large as the number of steps from a given point to the boundary.

Secondly, if we consider the sequence of values appropriate to a particular mesh point at the end of each relaxation, we find that they oscillate, those corresponding to a 'diagonal' grid forming one sequence, while those corresponding to a 'square' grid form a second sequence.

Comment on this type of oscillation in relation to the buckling of beams has been made by Salvadori (**5**), but no explanation is offered. We believe we have a partial explanation from a consideration of one of the simpler cases.

We shall show that, if the density function in Poisson's equation is harmonic, the two sequences converge to the solution in the continuum, one from above and one from below. Moreover, we suggest, there are many cases in which it is unnecessary to follow the usual procedure of solving a series of finer and finer meshes until no change is observed in the pivotal values. An extrapolation technique can be used to estimate the grid size which will be required for a given degree of accuracy, and some pivotal values can be accurately determined before relaxation is started on a fine grid if the overall solution of the problem is required.

2. Analysis

We may write Poisson's equation in the form

$$\nabla^2\phi(x,y)+\rho(x,y) = 0, \tag{1}$$

where $\phi(x,y)$ is the potential and $\rho(x,y)$ the density; when $\rho(x,y)$ is identically zero, we have Laplace's equation. We may set up the corresponding functional equation

$$u(x+h,y)+u(x-h,y)+u(x,y+h)+u(x,y-h)- \\ -4u(x,y)+h^2\rho(x,y) = 0, \tag{2}$$

for a grid with sides parallel to the coordinate axes. Here $u(x,y)$ corresponds to $\phi(x,y)$ and is a generalization of the point function used in the normal difference-equation, being identical with it if (x,y) is restricted to the grid points only. Clearly $u(x,y)$ is also dependent on the mesh size h.

For a 'diagonal' grid of mesh size h we have the functional equation

$$v(x+h/\surd 2,y+h/\surd 2)+v(x+h/\surd 2,y-h/\surd 2)+v(x-h/\surd 2,y+h/\surd 2)+ \\ +v(x-h/\surd 2,y-h/\surd 2)-4v(x,y)+h^2\rho(x,y) = 0. \tag{3}$$

Here $v(x,y)$ corresponds to the potential in equation (1).

We shall assume that the boundary conditions are that $\phi(x,y)$ is given on the boundary, and that this can be applied unambiguously to the functional equations. Clearly this will require modification if the grids

used do not conform to the boundary, and this can give rise to a difficulty which we shall discuss later.

Little seems to be known from the pure-mathematical standpoint about the nature of such functional equations in two dimensions, so we assume that solutions exist which are regular functions of x, y, h for values of h sufficiently small. Under these circumstances we can expand, where necessary, using a Taylor series and write, following the notation of Woods (**8**) and Bickley (**1**),

$$\left[\nabla^2+\frac{h^2}{12}(\nabla^4-2\mathscr{D}^4)+\frac{h^4}{360}(\nabla^6-3\mathscr{D}^4\nabla^2)+O(h^6)\right]u(x,y)+\rho(x,y) = 0, \quad (4)$$

and

$$\left[\nabla^2+\frac{h^2}{24}(\nabla^4+4\mathscr{D}^4)+\frac{h^4}{1440}(\nabla^6+12\mathscr{D}^4\nabla^2)+O(h^6)\right]v(x,y)+\rho(x,y) = 0, \quad (5)$$

where
$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}, \qquad \mathscr{D}^2 \equiv \frac{\partial^2}{\partial x\partial y}. \quad (6)$$

The functional difference-equations (2) and (3) have thus been replaced by differential equations, similar to the original differential equation (1), but involving correction terms which differ in the two cases of the 'square' and 'diagonal' grids. As has been mentioned by one of us elsewhere (**2**), there is a danger that solutions of differential equations of degree higher than the second may creep into the solution here, but this has yet to be observed in practice for what is essentially a jury problem.

Since only h^2 is really involved in the equations, we may suppose that $u(x,y)$ and $v(x,y)$ can be expanded in a power series in h^2, say

$$\left.\begin{aligned} u(x,y) &= \phi_0(x,y)+h^2\phi_1(x,y)+h^4\phi_2(x,y)+\ldots \\ v(x,y) &= \psi_0(x,y)+h^2\psi_1(x,y)+h^4\psi_2(x,y)+\ldots \end{aligned}\right\}. \quad (7)$$

Substituting into the differential equations (4) and (5) and comparing coefficients of powers of h^2, we see immediately that $\phi_0(x,y)$ and $\psi_0(x,y)$ must both be the required solution $\phi(x,y)$ of the original equation (1). Moreover, $\phi_1(x,y)$ and $\psi_1(x,y)$ are given by

$$\left.\begin{aligned} \nabla^2\phi_1(x,y) &= -\tfrac{1}{12}(\nabla^4-2\mathscr{D}^4)\phi(x,y) \\ \nabla^2\psi_1(x,y) &= -\tfrac{1}{24}(\nabla^4+4\mathscr{D}^4)\phi(x,y) \end{aligned}\right\}, \quad (8)$$

respectively, where the boundary condition on both ϕ_1 and ψ_1 is that they vanish on the boundary, leaving no ambiguity as to the solutions required. In certain special circumstances these principal correction terms are equal and opposite; they are, at the moment, corrections to

the solution of (1) such that equations (2) and (3) are satisfied. This is the case when $\nabla^2\rho(x, y) = 0$, implying $\nabla^4\phi = 0$. This is, of course, also true for Laplace's equation, i.e. when $\rho \equiv 0$. The second-correction terms (i.e. the h^4 terms) can be shown to be approximately equal when the density is harmonic, and equal when the density is identically zero.

From this we are led to suppose that, when the finite-difference equations corresponding to the functional equations (2) and (3) are solved with the correct insertion of the boundary conditions, then the pivotal values w_i at a particular point (denoted by i), considered as a function of the mesh size $h = 1/n$, can be approximately represented by

$$w_i(n) = A_i \pm B_i h^2 + C_i h^4 \pm \ldots, \tag{9}$$

where the coefficients depend on the particular lattice site and the alternative signs are taken positive for 'square' grids and negative for 'diagonal' grids.

In the case of Laplace's equation this hypothesis is correct, and, when the density function is harmonic, at least the first two terms are correct for Poisson's equation. For the general density function, the usefulness of the hypothesis will depend on the relative magnitudes of $\nabla^2\rho$ and $\mathscr{D}^4\phi$, as can be seen from equation (8).

The order of magnitude of B_i can be estimated by an approximate integration of equation (8) using data obtained from a coarse mesh, and this can be used, if required, to estimate the value of the mesh size h necessary for the difference between the finite-difference solution and the continuum solution to be negligible.

The validity of the hypothesis embodied in (9) is subject to the fairly severe restriction that the boundary has in some way to conform to the grid. There must be no ambiguity as to how the boundary conditions imposed on the potential $\phi(x, y)$ in (1) are to be applied to the difference equations, as distinct from the functional equations (2), (3). It is preferable that the boundary points of the grid should lie on the boundary curve.

That the restriction is very necessary is shown by considering the special case where the boundary is a circle. Clearly, in this case, there is no distinction between 'square' and 'diagonal' grids, so that no oscillation should be observed. However, the way in which the boundary conditions are applied to the finite-difference equations will modify the problem, and variations in the solutions as h^2 is decreased will depend on the way the boundary conditions are applied and not on changes due to grid orientation.

Thus we must restrict attention, for the strict application of (9), to cases where the oscillations of the solutions, as h^2 decreases, are due solely to changes in the grid orientation, and not to changes in the manner in which the boundary conditions are applied.

3. Example

The problem of the solution of Poisson's equation inside a square for constant density and zero boundary value has been considered by several authors, (**3**), (**8**), and an algebraic expression for the potential is known, (**6**), (**7**).

We have considered the problem of finding the potential at the centre of the square to an accuracy of 1 part in 10^4 following the methods discussed above.

Taking the differential equation [cf. Hartree (**4**) Ex. 37] in the form

$$\nabla^2 V = -2/a^2,$$

inside a square of side a, and $V = 0$ on the boundary of the square, the difference equation takes the form

$$\sum_j w_j - 4w_i + 2n^{-2} = 0.$$

Here w_i represents a typical pivotal value corresponding to either type of grid, $\sum_j w_j$ denotes the sum of values at the four immediate neighbours of i, and $n = ah^{-1}$, h being the mesh size.

Since we wish, in this instance, merely to examine the values of w corresponding to the centre of the square, only a restricted class of grids need be considered. For 'square' grids, n^2 is the number of small squares into which the basic square is divided, and n is given only the values 2, 4, 6, 8,.... For 'diagonal' grids, $n/\sqrt{2} = 1, 2, 3, 4,...$ give grid points at the centre.

The values of w corresponding to the centre of the square, and the various values of n, are shown in Table 1.

TABLE 1

n	2	4	6	8
w	0·12500	0·14062	0·14425	0·14560
n	$\sqrt{2}$	$2\sqrt{2}$	$3\sqrt{2}$	$4\sqrt{2}$
w	0·25	0·16667	0·15476	0·15127

For $n = 4\sqrt{2}$, 6, 8 the difference equations were solved by relaxation, the other cases being solved analytically.

The extrapolation technique used elsewhere by one of us (**2**) for extrapolating the eigenvalues of sets of difference equations to the case

of zero mesh size can be used here for an extrapolation to $h^2 = 0$ based on equation (9).

If $w(m)$ and $w(n)$ correspond to n-values m and n, then the 'linear' extrapolant $w(m, n)$ is given by

$$w(m, n) = \frac{w(n)n^2 \mp w(m)m^2}{n^2 \mp m^2},$$

the negative sign being taken if m and n correspond to the same grid type, and the positive sign being taken if m and n correspond to different types of grid.

This modification of a Neville extrapolation [(**4**) 85] can be extended quite straightforwardly to quadratic and cubic extrapolation, and a Neville table can be set up to display the extrapolants. Table 2 shows the extrapolants for the natural sequence of grids described in our introduction, while Table 3 shows the extrapolation for the four finest grids considered.

TABLE 2

n	$w(n)$	$w(m, n)$	$w(l, m, n)$
$\sqrt{2}$	0·25000		
		·16667	
2	0·12500		·14815
		·15278	
$2\sqrt{2}$	0·16667		·14814
		·14930	
4	0·14062		·14719
		·14772	
$4\sqrt{2}$	0·15127		·14741
		·14749	
8	0·14560		

TABLE 3

n	$w(n)$	$w(m, n)$	$w(l, m, n)$	$w(k, l, m, n)$
$3\sqrt{2}$	0·15476			
		·14775		
6	0·14425		·14729	
		·14755		·14738
$4\sqrt{2}$	0·15127		·14741	
		·14749		
8	0·14560			

The relaxation process leaves a significant rounding error, which in the case of $n = 8$ may be as much as 4 in the last significant figure, and, since the extrapolation technique tends to emphasize such small errors, we draw the conclusion from these two Neville tables that, in the continuum, the potential at the centre is $0{\cdot}1474 \pm 0{\cdot}0001$, which is in

agreement with the value 0·14734 obtained from the analytic solution and the value 0·1473 found by Fox (**3**) (after scaling).

The extrapolation technique applies equally well to other grid points which appear in several grids, the 'square' grids giving lower and 'diagonal' grids higher values than that corresponding to the continuum.

4. Conclusion

We have shown that the pivotal values at a particular point may be approximately represented by

$$w = A \pm Bh^2 + O(h^4),$$

where A and B depend on the position of the point and the alternative sign is taken when a 'diagonal' mesh is used, as compared with a 'square' mesh.

The approximation will be perfectly satisfactory if the density function ρ is harmonic, and will still be useful if $\nabla^2\rho$ is small compared with $\mathscr{D}^4\phi$. The main restrictive feature is that the change from 'square' to 'diagonal' grid should not affect the way in which the boundary conditions are applied, and so will have straightforward application only for problems with rectangular boundaries.

The advantage of the method is that in a particular problem the value of the potential at a few points can be fixed with some accuracy, and these can be used to obtain a more detailed picture of the solution using either a fine mesh, or the more sophisticated methods of Fox (**3**). In this way many of the difficulties of the relaxation process for fine meshes can be avoided, especially if computing facilities are available for solving accurately the finite-difference equations for the smaller values of h, where the number of grid points is not more than (say) 40.

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FOURIER SERIES WITH GAPS

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1. Let $\{n_k\}$ $(k = 1, 2, 3,...)$ be a strictly increasing sequence of positive integers. In this note $f(x)$ will denote a real function, L-integrable over $(-\pi, \pi)$ and having period 2π, whose Fourier coefficients a_n, b_n vanish except possibly when n has any of the values n_k. Noble (**1**) supposed that

$$\frac{n_{k+1}-n_k}{\log n_k} \to \infty \quad (k \to \infty), \tag{1.1}$$

and derived a number of properties of the coefficients a_n, b_n under various hypotheses about the behaviour of $f(x)$ in an arbitrarily small interval. Noble's proofs start from the fact that

$$a_{n_k} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)P(x)\cos n_k x\,dx$$

(with a similar formula for b_{n_k}), for any trigonometrical polynomial $P(x)$, with constant term 1, of degree less than $\min(n_k-n_{k-1}, n_{k+1}-n_k)$. He shows that, given α and β with $-\pi < \alpha < \beta < \pi$, one can choose $P(x)$ so as to be very small in $(-\pi, \alpha)$ and (β, π); $|a_{n_k}|$ is thus made to depend mainly on the behaviour of $f(x)P(x)$ in (α, β).

More powerful methods of approach to this kind of problem were developed by Paley and Wiener (**2**). In this note I show how these methods can be used to give quite simple proofs of Noble's results, under the less restrictive gap hypothesis

$$n_{k+1}-n_k \to \infty \quad (k \to \infty). \tag{1.2}$$

2. Following Paley and Wiener (**2**), we consider, in the first instance, a more general situation than that described in § 1. Let

$$\{\lambda_k\} \quad (-\infty < k < \infty)$$

be a sequence of real numbers satisfying

$$\lambda_{k+1} > \lambda_k \quad (-\infty < k < \infty), \qquad \lambda_{k+1}-\lambda_k \to \infty \quad (|k| \to \infty), \tag{2.1}$$

and let $\{A_k\}$ $(-\infty < k < \infty)$ be a sequence of complex numbers such that

$$\sum_{-\infty}^{\infty} |A_k| r^{|\lambda_k|} < \infty \quad (0 < r < 1). \tag{2.2}$$

Put
$$\phi(r,x) = \sum_{-\infty}^{\infty} A_k r^{|\lambda_k|} e^{i\lambda_k x} \quad (0 < r < 1) \tag{2.3}$$

for all real x; the existence of $\phi(r,x)$ is ensured by (2.2).

Suppose now that

$$\phi(x) = L^2\text{-}\lim_{r\to 1} \phi(r,x) \quad (|x-x_0| \leqslant \delta), \tag{2.4}$$

where x_0 is fixed and $\delta > 0$. We then have the following theorems.

THEOREM I. *Suppose that $\phi(x)$ has bounded variation in $|x-x_0| \leqslant \delta$. Then*

$$A_k = O(|\lambda_k|^{-1}) \quad (|k| \to \infty).$$

THEOREM II. *Suppose that $\phi(x) \in \operatorname{Lip}\alpha$ in $|x-x_0| \leqslant \delta$, where*

$$0 < \alpha < 1.$$

Then
$$A_k = O(|\lambda_k|^{-\alpha}) \quad (|k| \to \infty).$$

THEOREM III. *Suppose that $\phi(x) \in \operatorname{Lip}\alpha$ in $|x-x_0| \leqslant \delta$, where*

$$\tfrac{1}{2} < \alpha < 1.$$

Then
$$\sum_{-\infty}^{\infty} |A_k| < \infty. \tag{2.5}$$

THEOREM IV. *Suppose that, in $|x-x_0| \leqslant \delta$, $\phi(x)$ has bounded variation and $\phi(x) \in \operatorname{Lip}\alpha$, where $0 < \alpha < 1$. Then* (2.5) *is true.*

Returning to the situation described in § 1, we deduce from Theorems I–IV the following theorem on Fourier series with gaps, which comprises Theorems 1, 2, 3, and 5 of (**1**), with (1.1) replaced by (1.2).

THEOREM V. *Let $\{n_k\}$ and $f(x)$ be as in § 1, and suppose that* (1.2) *holds. Then the following statements are true.*

(i) *If $f(x)$ has bounded variation in some interval I, then*

$$a_n, b_n = O(n^{-1}) \quad (n \to \infty).$$

(ii) *If $f(x) \in \operatorname{Lip}\alpha$ in some interval I, where $0 < \alpha < 1$, then*

$$a_n, b_n = O(n^{-\alpha}) \quad (n \to \infty).$$

(iii) *If $f(x) \in \operatorname{Lip}\alpha$ in some interval I, where $\frac{1}{2} < \alpha < 1$, then*

$$\sum_{1}^{\infty} (|a_n| + |b_n|) < \infty. \tag{2.6}$$

(iv) *If, in some interval I, $f(x)$ has bounded variation and $f(x) \in \operatorname{Lip}\alpha$, where $0 < \alpha < 1$, then* (2.6) *is true.*

Without the hypothesis (1.2), and with $(-\pi,\pi)$ for I, Theorem V is classical [(**3**), 18, 135].

3. The key to the proofs of Theorems I–IV is the following lemma, which is essentially due to Paley and Wiener (**2**).

LEMMA 1. *Let* $\{\lambda_k\}$, $\{A_k\}$, $\phi(x)$, x_0, *and* δ *be as in* § 2, *but let* (2.1) *be replaced by*

$$\lambda_{k+1}-\lambda_k > 8\pi\delta^{-1} \quad (-\infty < k < \infty). \tag{3.1}$$

Then

$$\sum_{-\infty}^{\infty} |A_k|^2 \leqslant 8\delta^{-1} \int_{|x-x_0|\leqslant\delta} |\phi(x)|^2\,dx.$$

To prove Lemma 1, put

$$\phi(r,x;K) = \sum_{-K}^{K} A_k r^{|\lambda_k|} e^{i\lambda_k x} \quad (0 < r < 1). \tag{3.2}$$

Then (3.1) and the argument by which Paley and Wiener proved the inequality (31.10) of (**2**) lead to

$$\int_{|x-x_0|\leqslant\delta} |\phi(r,x;K)|^2 \frac{\sin^2\{4\pi\delta^{-1}(x-x_0)\}}{(x-x_0)^2}\,dx \geqslant 2\pi^2\delta^{-1} \sum_{-K}^{K} |A_k|^2 r^{2|\lambda_k|},$$

and this, with the inequality

$$1 \geqslant \frac{\delta^2 \sin^2\{4\pi\delta^{-1}(x-x_0)\}}{16\pi^2(x-x_0)^2},$$

gives

$$\int_{|x-x_0|\leqslant\delta} |\phi(r,x;K)|^2\,dx \geqslant \tfrac{1}{8}\delta \sum_{-K}^{K} |A_k|^2 r^{2|\lambda_k|}. \tag{3.3}$$

But it follows from (2.2), (2.3), (3.2) that, for any fixed r such that $0<r<1$,

$$\phi(r,x;K) \to \phi(r,x) \quad (K\to\infty)$$

uniformly in every finite range of values of x. We therefore deduce from (3.3) that

$$\int_{|x-x_0|\leqslant\delta} |\phi(r,x)|^2\,dx \geqslant \tfrac{1}{8}\delta \sum_{-\infty}^{\infty} |A_k|^2 r^{2|\lambda_k|}.$$

Lemma 1 now follows since (2.4) implies

$$\int_{|x-x_0|\leqslant\delta} |\phi(r,x)|^2\,dx \to \int_{|x-x_0|\leqslant\delta} |\phi(x)|^2\,dx \quad (r\to 1).$$

The deduction of Theorems I–IV from Lemma 1 proceeds by fairly straightforward generalization of the arguments used to prove the classical analogue of Theorem V; the part played by Bessel's inequality in the classical theory is played in our proofs by Lemma 1. We assume throughout that

$$\lambda_{k+1}-\lambda_k > 16\pi\delta^{-1} \quad (-\infty < k < \infty). \tag{3.4}$$

In view of (2.1), this can be achieved, if necessary, by adding to $\phi(x)$ a polynomial in $e^{i\lambda_k x}$, a process which affects neither the hypotheses nor the conclusions of the theorems.

From Lemma 1 we deduce the following lemma, which we need for the proof of Theorems II, III, and IV.

LEMMA 2. *Let $\{\lambda_k\}$, $\{A_k\}$, $\phi(x)$, x_0, and δ be as in § 2, but let (2.1) be replaced by (3.4). Let the integers j, K satisfy*

$$|\lambda_K| > 2\pi\delta^{-1}, \qquad 0 \leqslant j \leqslant \tfrac{1}{4}\pi^{-1}\delta|\lambda_K|. \tag{3.5}$$

Put
$$\phi_j(x) = \phi\left\{x+\frac{2j\pi}{|\lambda_K|}\right\}-\phi\left\{x+\frac{(2j-1)\pi}{|\lambda_K|}\right\}. \tag{3.6}$$

Then
$$S_K \equiv \sum_{\frac{1}{2}|\lambda_K|\leqslant|\lambda_k|\leqslant|\lambda_K|} |A_k|^2 \leqslant 8\delta^{-1} \int\limits_{|x-x_0|\leqslant\frac{1}{2}\delta} |\phi_j(x)|^2\,dx. \tag{3.7}$$

To prove Lemma 2, put

$$A_k(j) = 2iA_k \exp\left\{\frac{i(2j-\frac{1}{2})\pi\lambda_k}{|\lambda_K|}\right\}\sin\frac{\pi\lambda_k}{2|\lambda_K|} \quad (-\infty < k < \infty). \tag{3.8}$$

Since $|A_k(j)| \leqslant 2|A_k|$, we have, by (2.2),

$$\sum_{k=-\infty}^{\infty} |A_k(j)|r^{|\lambda_k|} < \infty \quad (0 < r < 1). \tag{3.9}$$

Put
$$\phi_j(r,x) = \sum_{k=-\infty}^{\infty} A_k(j)r^{|\lambda_k|}e^{i\lambda_k x} \quad (0 < r < 1). \tag{3.10}$$

The identity

$$\phi_j(r,x) = \phi\left\{r,\, x+\frac{2j\pi}{|\lambda_K|}\right\}-\phi\left\{r,\, x+\frac{(2j-1)\pi}{|\lambda_K|}\right\}$$

is easily verified, and from it, combined with (2.4) and (3.6), we obtain

$$\phi_j(x) = L^2\text{-}\lim_{r\to 1}\phi_j(r,x) \quad (|x-x_0| \leqslant \tfrac{1}{2}\delta), \tag{3.11}$$

since, by (3.5), the points

$$x+(2j-1)\pi|\lambda_K|^{-1}, \qquad x+2j\pi|\lambda_K|^{-1}$$

lie in the interval $(x_0-\delta, x_0+\delta)$ whenever $|x-x_0| \leqslant \frac{1}{2}\delta$. It now follows from (3.4), (3.9), (3.10), (3.11) and Lemma 1 (with A_k, ϕ, δ replaced by $A_k(j)$, ϕ_j, $\frac{1}{2}\delta$) that

$$\sum_{k=-\infty}^{\infty} |A_k(j)|^2 \leqslant 16\delta^{-1} \int\limits_{|x-x_0|\leqslant\frac{1}{2}\delta} |\phi_j(x)|^2\,dx. \tag{3.12}$$

But by (3.8) and the obvious inequality

$$\sin^2\frac{\pi\lambda_k}{2|\lambda_K|} \geqslant \tfrac{1}{2} \quad (\tfrac{1}{2}|\lambda_K| \leqslant |\lambda_k| \leqslant |\lambda_K|),$$

we have
$$\sum_{k=-\infty}^{\infty} |A_k(j)|^2 \geqslant 2S_K,$$
where S_K is as in (3.7). This, with (3.12), gives (3.7), and so Lemma 2 is proved.

4. Proof of Theorems I–IV

Throughout this section the integer K is assumed to satisfy the first inequality of (3.5), and the symbol $\sum_j$ denotes summation over all integers j satisfying the second inequality of (3.5).

For the proof of Theorem I we do not use Lemma 2, but employ the notations introduced in that lemma and in its proof. Put
$$A_k^* = \sum_j A_k(j) \quad (-\infty < k < \infty). \tag{4.1}$$
Then
$$\sum_{-\infty}^{\infty} |A_k^*| r^{|\lambda_k|} < \infty \quad (0 < r < 1). \tag{4.2}$$
Put
$$\phi^*(r,x) = \sum_{-\infty}^{\infty} A_k^* r^{|\lambda_k|} e^{i\lambda_k x} = \sum_j \phi_j(r,x), \tag{4.3}$$
so that, by (3.11),
$$L^2\text{-}\lim_{r\to 1} \phi^*(r,x) = \sum_j \phi_j(x) \equiv \phi^*(x), \text{ say}, \tag{4.4}$$
over the range $|x-x_0| \leqslant \frac{1}{2}\delta$. It follows from (3.4), (4.2), (4.3), (4.4), and Lemma 1 (with A_k, ϕ, δ replaced by A_k^*, ϕ^*, $\frac{1}{2}\delta$) that
$$|A_K^*|^2 \leqslant \sum_{-\infty}^{\infty} |A_k^*|^2 \leqslant 16\delta^{-1} \int_{|x-x_0|\leqslant\frac{1}{2}\delta} |\phi^*(x)|^2\, dx. \tag{4.5}$$

From (3.8) we have
$$A_K(j) = 2A_K \quad (0 \leqslant j \leqslant \tfrac{1}{4}\pi^{-1}\delta|\lambda_K|),$$
and so, by (4.1),
$$|A_K^*|^2 \geqslant \tfrac{1}{4}\pi^{-2}\delta^2|\lambda_K|^2|A_K|^2. \tag{4.6}$$
Also, for $|x-x_0| \leqslant \frac{1}{2}\delta$ and integers j, K satisfying (3.5), the intervals
$$\{x+(2j-1)\pi|\lambda_K|^{-1},\ x+2j\pi|\lambda_K|^{-1}\}$$
are non-overlapping subintervals of $(x_0-\delta, x_0+\delta)$. Therefore, if V is the total variation of $\phi(x)$ in $|x-x_0| \leqslant \delta$, we have
$$|\phi^*(x)| \leqslant \sum_j |\phi_j(x)| \leqslant V \quad (|x-x_0| \leqslant \tfrac{1}{2}\delta).$$
This, combined with (4.5) and (4.6), gives
$$|A_K|^2 = O(|\lambda_K|^{-2}) \quad (|K| \to \infty).$$
This proves Theorem I.

To prove Theorem II, we apply Lemma 2 with any fixed integer j

satisfying (3.5). We observe that $S_K \geqslant |A_K|^2$, while the right-hand side of (3.7) is at most equal to

$$8\left\{\omega\left(\frac{\pi}{|\lambda_K|}\right)\right\}^2,$$

where $\omega(t)$ is the modulus of continuity of $\phi(x)$ in $(x_0-\delta, x_0+\delta)$. Therefore, by the Lipschitz condition,

$$|A_K|^2 = O(|\lambda_K|^{-2\alpha}) \quad (|K| \to \infty).$$

This proves Theorem II.

To prove Theorem III we start with

$$S_K = O(|\lambda_K|^{-2\alpha}) \quad (|K| \to \infty),$$

which we obtain from (3.7) just as in the preceding paragraph. Let p be a positive integer. Either the set of integers k for which

$$2^p < |\lambda_k| \leqslant 2^{p+1}$$

is empty, or there is a member of this set, say $K = K(p)$, which has largest modulus; and in the latter case the set is included in the set of k for which $\frac{1}{2}|\lambda_K| \leqslant |\lambda_k| \leqslant |\lambda_K|$. Thus, in either case,

$$\sum_{2^p<|\lambda_k|\leqslant 2^{p+1}} |A_k|^2 \leqslant S_K = O(|\lambda_K|^{-2\alpha}) = O(2^{-2\alpha p}).$$

From this we obtain (2.5) by a familiar application of Cauchy's inequality. In fact, by (3.4), the number of terms in the summation is $O(2^p)$, and so, by Cauchy's inequality,

$$\sum_{2^p<|\lambda_k|\leqslant 2^{p+1}} |A_k| = O\left\{\left(2^p \sum_{2^p<|\lambda_k|\leqslant 2^{p+1}} |A_k|^2\right)^{\frac{1}{2}}\right\} = O\{2^{(\frac{1}{2}-\alpha)p}\}.$$

Now (2.5) follows at once, since $\alpha > \frac{1}{2}$. This proves Theorem III.

To prove Theorem IV we again apply Lemma 2. By the hypotheses of Theorem IV and the sentence immediately following (4.6), we have

$$\sum_j |\phi_j(x)|^2 \leqslant V\omega\left(\frac{\pi}{|\lambda_K|}\right) = O(|\lambda_K|^{-\alpha}),$$

uniformly for $|x-x_0| \leqslant \frac{1}{2}\delta$. Therefore, summing (3.7) over

$$0 \leqslant j \leqslant \tfrac{1}{4}\pi^{-1}\delta|\lambda_K|,$$

we obtain
$$\tfrac{1}{4}\pi^{-1}\delta|\lambda_K|S_K = O(|\lambda_K|^{-\alpha}),$$

and so $S_K = O(|\lambda_K|^{-1-\alpha})$. We can now deduce (2.5) from this by arguing with Cauchy's inequality, as in the proof of Theorem III. This proves Theorem IV.

5. Proof of Theorem V

To prove Theorem V, let $\{n_k\}$, $f(x)$, a_n, and b_n be as in § 1. Put

$$A_0 = 0, \qquad A_k = \tfrac{1}{2}(a_{n_k} - ib_{n_k}) \quad (k > 0), \qquad A_k = \bar{A}_{-k} \quad (k < 0),$$

and
$$\lambda_k = n_k \quad (k > 0), \qquad \lambda_k = -\lambda_{-k} \quad (k < 0).$$

Then (2.1) and (2.2) are satisfied, and

$$\phi(r, x) = \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n.$$

Theorem V now follows at once from Theorems I–IV, with $\phi(x) = f(x)$, by virtue of the following simple lemma.

LEMMA 3. *If $f(x) \in L^2(I)$ and $|x - x_0| \leqslant \delta$ is a proper subinterval of I, then*

$$f(x) = L^2\text{-}\lim_{r \to 1} \phi(r, x) \quad (|x - x_0| \leqslant \delta).$$

This lemma is almost certainly known, but I indicate the proof since I cannot quote a source.

We suppose, as we may without loss of generality, that I is a subinterval of $(-\pi, \pi)$. Put

$$f_1(x) = f(x) \quad (x \in I), \qquad f_1(x) = 0 \quad \text{otherwise},$$
$$f_2(x) = f(x) - f_1(x).$$

Let $a_n^{(1)}$, $b_n^{(1)}$ and $a_n^{(2)}$, $b_n^{(2)}$ be the Fourier coefficients of $f_1(x)$ and $f_2(x)$ respectively, and, for $j = 1, 2$, let

$$\phi^{(j)}(r, x) = \tfrac{1}{2} a_0^{(j)} + \sum_{n=1}^{\infty} (a_n^{(j)} \cos nx + b_n^{(j)} \sin nx) r^n \quad (0 < r < 1).$$

Then, since $f_1(x) \in L^2(-\pi, \pi)$, it follows from a known theorem [(**3**) 87] that

$$f_1(x) = L^2\text{-}\lim_{r \to 1} \phi^{(1)}(r, x) \quad (|x| \leqslant \pi). \tag{5.1}$$

Further, by Fejér's theorem on A-summability of a Fourier series [(**3**) 51], we have

$$\phi^{(2)}(r, x) \to f_2(x) \quad (r \to 1), \tag{5.2}$$

uniformly for $|x - x_0| \leqslant \delta$. Since

$$\phi(r, x) = \phi^{(1)}(r, x) + \phi^{(2)}(r, x),$$

Lemma 3 follows from (5.1) and (5.2). This proves Theorem V.

I wish to thank the referee for his suggestions.

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ON AN EXPANSION IN EXPONENTIAL SERIES

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1. Introduction

LET $k(u)$ be a function of bounded variation, possibly complex, defined in the interval $(0,1)$. We shall be concerned with the expansion of a function in a series of exponentials $e^{\lambda_\nu t}$, where the λ_ν are the zeros of the integral function

$$A(z) = \int_0^1 k(u)e^{zu}\,du. \tag{1}$$

We may suppose that

$$k(0) = k(0+), \qquad k(1) = k(1-). \tag{2}$$

The method of investigation is based on the theory of residues, and requires some information about the order of magnitude of $A(z)$ on the path of integration. This is attained by restricting $k(u)$ to satisfy the condition

$$k(1)k(0) \neq 0. \tag{3}$$

I shall suppose that all the zeros of $A(z)$ are simple, i.e. that

$$A'(\lambda_\nu) \neq 0, \tag{4}$$

and also that

$$A(0) \neq 0. \tag{5}$$

Conditions (4) and (5) could be relaxed without difficulty. The formulae would then become complicated. Choose the notation so that

$$0 < |\lambda_1| \leqslant |\lambda_2| \leqslant \dots .$$

Let

$$\phi_0(u) = k(u), \qquad \phi_\nu(u) = \frac{k(u)}{\lambda_\nu} + e^{-\lambda_\nu u}\int_u^1 k(v)e^{\lambda_\nu v}\,dv, \tag{6}$$

$$\psi_0(u) = 1, \qquad \psi_\nu(u) = e^{\lambda_\nu u}, \tag{7}$$

where $\nu = 1, 2,\dots$. Then the functions ϕ, ψ are bi-orthogonal in $(0,1)$, and

$$\int_0^1 \phi_0\psi_0\,du = A(0), \qquad \int_0^1 \phi_\nu\psi_\nu\,du = A'(\lambda_\nu). \tag{8}$$

For a function $f(u) \in L(0,1)$ we write

$$\alpha_0 = \frac{1}{A(0)} \int\limits_0^1 f(u)\phi_0(u)\,du, \tag{9}$$

$$\alpha_\nu = \frac{1}{A'(\lambda_\nu)} \int\limits_0^1 f(u)\phi_\nu(u)\,du \quad (\nu = 1, 2,...). \tag{10}$$

Let
$$B(z) = zA(z). \tag{11}$$

Then
$$B(z) = k(1)e^z - k(0) - \int\limits_0^1 e^{zu}\,dk(u). \tag{12}$$

By (2) and (3) it follows that there is a positive constant C such that

$$|B(z)| > \begin{cases} \frac{1}{2}|k(1)e^z| & (x \geqslant C), \\ \frac{1}{2}|k(0)| & (x \leqslant -C), \end{cases} \tag{13}$$

where $z = x+iy$. The zeros of $B(z)$ are therefore in the strip $|x| \leqslant C$, and a simple argument [see Lemma 2, § 2] shows that the number of zeros of $B(z)$ in the strip

$$2n\pi \leqslant y \leqslant (2n+2)\pi$$

is bounded with respect to the integer n. We can therefore find a positive number δ such that, if each zero of $B(z)$ is the centre of a disk of radius δ, then there is an unbounded increasing sequence of positive numbers r_p such that the circle C_p: $|z| = r_p$ has no points in common with the disks. There is a wide choice for the r_p: e.g. they can be chosen so that $r_{p+1}-r_p \leqslant 1$. Let ν_p denote the greatest integer ν such that $|\lambda_\nu| < r_p$. I shall prove

THEOREM 1. *Let $f(u) \in L(0,1)$. Then, as $p \to \infty$,*

$$\alpha_0 + \sum_1^{\nu_p} \alpha_\nu e^{\lambda_\nu t} - \frac{1}{\pi} \int\limits_0^1 f(u) \frac{\sin r_p(t-u)}{t-u}\,du$$

converges to zero uniformly in any closed interval (α, β) interior to the open interval $(0,1)$.

An easy consequence is

THEOREM 2. *Let $f(u)$ be integrable in every finite interval, and satisfy the equation*

$$\int_0^1 k(u)f(t+u)\,du = 0 \tag{14}$$

for all t. Let (a,b) be an assigned finite interval. Then, as $p \to \infty$,

$$\sum_1^{\nu_p} \alpha_\nu e^{\lambda_\nu t} - \frac{1}{\pi}\int_a^b f(u)\frac{\sin r_p(t-u)}{t-u}\,du$$

converges to zero uniformly in any closed interval interior to the open interval (a,b).

A slight change in the proof (see § 5) shows that, if the set of distances between pairs of zeros of $A(z)$ has a positive lower bound, then we can put $\nu_p = p$ in Theorems 1 and 2, and replace r_p by $|\lambda_p|$.

The case of Theorem 1 in which $f(u)$ is of bounded variation in $(0,1)$ and satisfies the condition

$$\int_0^1 k(u)f(u)\,du = 0,$$

as also the case of Theorem 2 in which $f(u)$ is of bounded variation in every finite interval, has been considered by Delsarte (**1**). It should be remarked that Delsarte subjects $k(u)$ to a further restriction, viz. that the continuous part of $k(u)$ shall be absolutely continuous. The proof of Delsarte is rather long, and would seem to be incomplete. [See (**1**) 438, line 15.]

2. The function $B(z)$

It will be convenient to introduce the following

DEFINITION. *An integral function $H(z)$ will be said to have the 'δ-η property' on a set S if, given $\delta > 0$, there is a positive number η $(= \eta(\delta))$ such that, if each zero of $H(z)$ is the centre of a disk of radius δ, then, for all z in S outside the disks, $|H(z)| > \eta$.*

If S is the sum of sets S_j $(j = 1,\ldots, n)$ on each of which $H(z)$ has the δ-η property (with different functions $\eta(\delta)$), then $H(z)$ has the δ-η property on S.

LEMMA 1. *$B(z)$ has the δ-η property on the whole plane.*

Proof. By (13), $B(z)$ has the δ-η property on the set $x \geqslant C$ and on the set $x \leqslant -C$. It suffices to prove that $B(z)$ has the δ-η property

on the set $|x| \leqslant C$. If not, there is a $\delta > 0$ and a sequence of points ζ_ν $(= \xi_\nu + i\eta_\nu)$, with $|\xi_\nu| \leqslant C$ such that

$$|\eta_\nu| \to \infty, \qquad B(\zeta_\nu) \to 0, \qquad B(z) \neq 0$$

for $|z-\zeta_\nu| < \delta$ and $|x| \leqslant C$. The condition $|x| \leqslant C$ may here be omitted since $B(z) \neq 0$ for $|x| > C$. We may suppose that $\eta_\nu \to \infty$ or $-\infty$; say the former.

Let R_m denote the rectangle

$$|x| \leqslant C, \qquad 2m\pi \leqslant y \leqslant (2m+2)\pi \quad (m = 0, 1,...).$$

To each ν there corresponds an m_ν such that $\zeta_\nu \in R_{m_\nu}$. Write

$$\zeta'_\nu = \zeta_\nu - 2\pi i m_\nu.$$

We may suppose that the sequence ζ'_ν is convergent; say, $\zeta_\nu \to \zeta'$. Let

$$F_m(z) = B(z+2\pi i m) = k(1)e^z - k(0) - \int_0^1 e^{zu} e^{2\pi i m u}\, dk(u).$$

Write $G_\nu(z) = F_{m_\nu}(z)$. Then

$$G_\nu(\zeta'_\nu) \to 0, \tag{15}$$

$$G_\nu(z) \neq 0 \quad \text{for } |z-\zeta'_\nu| < \delta. \tag{16}$$

Consider the functions $G_\nu(z)$. They are uniformly bounded in the rectangle

$$R^*\colon |x| \leqslant C+1, \qquad -1 \leqslant y \leqslant 2\pi+1.$$

Let R' denote the rectangle

$$|x| \leqslant C+\tfrac{1}{2}, \qquad -\tfrac{1}{2} \leqslant y \leqslant 2\pi+\tfrac{1}{2}.$$

There is a sub-sequence $G_{\nu_r}(z)$ and a function $G(z)$ such that

(i) $\quad G_{\nu_r}(z) \to G(z)$ uniformly in R';

(ii) $\quad G_{\nu_r}^{(n)}(z) \to G^{(n)}(z)$ $(n = 1, 2,...)$ uniformly in R'.

By (15) and (i), $G(\zeta') = 0$. Now $G(z) \not\equiv 0$. For

$$|G_{\nu_r}^{(n)}(z)| = \left|k(1)e^z - \int_0^1 u^n e^{zu} e^{2\pi i m u}\, dk(u)\right| \quad (m = m_{\nu_r})$$

$$> \tfrac{1}{2}|k(1)|e^{-C}$$

for $|x| \leqslant C$ if n is sufficiently large. It follows that there is a zero of $G_{\nu_r}(z)$ in any assigned neighbourhood of ζ' if r is sufficiently large. This contradicts (16).

LEMMA 2. *The number of zeros of $B(z)$ in R_n $(n = 0, \pm 1,...)$ is bounded with respect to n.*

Proof. If not, there is a sequence m_ν $(\nu = 1, 2,...)$ with $|m_\nu| \to \infty$ such

that the number of zeros of $B(z)$ in R_{m_ν} tends to ∞. We may suppose that $m_\nu \to \infty$ or $-\infty$, say the former. Write

$$G_\nu = F_{m_\nu}, \qquad T_\nu = R_{m_\nu}.$$

There is a sub-sequence G_{ν_r} tending uniformly in R' to G ($\not\equiv 0$). Hence the number of zeros of G_{ν_r} in R_0 (i.e. the number of zeros of B in T_{ν_r}) does not exceed the number of zeros of G in R' if r is sufficiently large. This is a contradiction.

The contour C_p of § 1 will be traversed in the positive direction. The parts for which $x \geqslant 0$, $x < 0$ will be denoted by C_p^+, C_p^- respectively. By Lemma 1 and (13), we have, as $p \to \infty$,

$$\frac{e^z}{B(z)} = O(1) \quad \text{on } C_p^+, \tag{17}$$

$$\frac{1}{B(z)} = O(1) \quad \text{on } C_p^-. \tag{18}$$

3. Proof of Theorem 1

Write
$$g(t) = \int_0^1 k(u)f(t+u)\,du. \tag{19}$$

Consider the expression

$$\begin{aligned} I_p &= \frac{1}{2\pi i}\int_{C_p} \frac{e^{zt}}{A(z)}\,dz \int_0^1 k(u)e^{zu}\,du \int_0^u f(v)e^{-zv}\,dv \\ &= \frac{1}{2\pi i}\int_{C_p} \frac{e^{zt}}{A(z)}\,dz \int_0^1 f(v)e^{-zv}\,dv \int_v^1 k(u)e^{zu}\,du \\ &= \sum_1^{\nu_p} \frac{e^{\lambda_\nu t}}{A'(\lambda_\nu)} \int_0^1 f(v)\left\{\phi_\nu(v) - \frac{k(v)}{\lambda_\nu}\right\} dv \\ &= \sum_1^{\nu_p} \alpha_\nu e^{\lambda_\nu t} - g(0) \sum_1^{\nu_p} \frac{e^{\lambda_\nu t}}{\lambda_\nu A'(\lambda_\nu)}. \end{aligned} \tag{20}$$

We can write
$$\begin{aligned} 2\pi i I_p &= \int_{C_p^-} + \int_{C_p^+} \\ &= J_1 + J_2. \end{aligned} \tag{21}$$

Now
$$z\int_v^1 k(u)e^{zu}\,du = k(1)e^z - k(v)e^{zv} - \int_v^1 e^{zu}\,dk(u).$$

Hence

$$J_1 = \int_{C_p^-} \frac{e^{zt}}{B(z)}\,dz \int_0^1 f(v)e^{-zv}\left\{k(1)e^{z}-k(v)e^{zv}-\int_v^1 e^{zu}\,dk(u)\right\}dv$$

$$= \int_{C_p^-} \frac{k(1)}{B(z)}e^{zt}\,dz \int_0^1 f(v)e^{z(1-v)}\,dv - g(0)\int_{C_p^-} \frac{e^{zt}}{B(z)}\,dz -$$

$$-\int_{C_p^-} \frac{e^{zt}}{B(z)}\,dz \int_0^1 dk(u) \int_0^u f(v)e^{z(u-v)}\,dv$$

$$= K_1-K_2-K_3.$$

We recall that an integral of the form

$$\int_p^q \phi(u+\mu)e^{zu}\,du,$$

where μ, p, q are bounded parameters such that $q > p \geqslant 0$ and ϕ is L-integrable in the interval (min $p+\mu$, max $q+\mu$), tends to zero uniformly as $|z| \to \infty$ with $x \leqslant 0$. Further,

$$\int_{C_p^-} |e^{\lambda z}\,dz| = O(1)$$

as $p \to \infty$ if λ is a positive constant, and is uniformly $O(1)$ if λ is a parameter bounded below by a positive constant.

Consider K_1. On writing $1-v = w$, we see that

$$\int_0^1 f(v)e^{z(1-v)}\,dv$$

is uniformly $o(1)$ on C_p^-. Now $t \geqslant \alpha > 0$. By (18), K_1 is uniformly $o(1)$.

Consider K_3. Then

$$M(u) = \int_0^u f(v)e^{z(u-v)}\,dv$$

is uniformly $o(1)$ on C_p^-. So then is

$$\int_0^1 M(u)\,dk(u).$$

Hence, as before, K_3 is uniformly $o(1)$.

Next

$$J_2 = \int_{C_p^+} \frac{e^{zt}}{A(z)}\, dz \int_0^1 f(v)e^{-zv} \left[\int_0^1 - \int_0^v\right] k(u)e^{zu}\, du dv$$

$$= \int_{C_p^+} e^{zt}\, dz \int_0^1 f(v)e^{-zv}\, dv - \int_{C_p^+} \frac{e^{zt}}{A(z)}\, dz \int_0^1 f(v)e^{-zv} \int_0^v k(u)e^{zu}\, du dv$$

$$= L_1 - L_2.$$

Then

$$L_1 = \int_0^1 f(v)\, dv \int_{C_p^-} e^{z(t-v)}\, dz = 2i \int_0^1 f(v) \frac{\sin r_p(t-v)}{t-v}\, dv.$$

Since
$$z\int_0^v k(u)e^{zu}\, du = k(v)e^{zv} - k(0) - \int_0^v e^{zu}\, dk(u),$$

we have

$$L_2 = \int_{C_p^+} \frac{e^{zt}}{B(z)}\, dz \int_0^1 f(v)e^{-zv}\left\{k(v)e^{zv} - k(0) - \int_0^v e^{zu}\, dk(u)\right\} dv$$

$$= g(0) \int_{C_p^+} \frac{e^{zt}}{B(z)}\, dz - M_1 - M_2,$$

where

$$M_1 = \int_{C_p^+} k(0) \frac{e^z}{B(z)} e^{-z(1-t)}\, dz \int_0^1 f(v)e^{-zv}\, dv,$$

$$M_2 = \int_{C_p^+} \frac{e^z}{B(z)} e^{-z(1-t)}\, dz \int_0^1 dk(u) \int_u^1 f(v)e^{z(u-v)}\, dv.$$

In M_1,
$$\int_0^1 f(v)e^{-zv}\, dv$$

is uniformly $o(1)$ on C_p^+. Now $1-t \geqslant 1-\beta > 0$. By (17), M_1 is uniformly $o(1)$. In M_2,

$$N(u) = \int_u^1 f(v)e^{z(u-v)}\, dv$$

is uniformly $o(1)$ on C_p^+. So then is

$$\int_0^1 N(u)\, dk(u).$$

Hence M_2 is uniformly $o(1)$. Thus

$$J_1+J_2 = L_1-g(0)\int_{C_p}\frac{e^{zt}}{B(z)}\,dz+\epsilon_p(t),$$

where $\epsilon_p(t)$ is uniformly $o(1)$. Since

$$\frac{1}{2\pi i}\int_{C_p}\frac{e^{zt}}{B(z)}\,dz = \frac{1}{A(0)}+\sum_{1}^{\nu_p}\frac{e^{\lambda_\nu t}}{\lambda_\nu A'(\lambda_\nu)},$$

and $\alpha_0 = g(0)/A(0)$, this proves the theorem.

With the notation

$$\alpha_\nu^* = \frac{1}{A'(\lambda_\nu)}\int_0^1 f(u)e^{-\lambda_\nu u}\,du\int_u^1 k(v)e^{\lambda_\nu v}\,dv,$$

we have the following

COROLLARY. *Under the conditions of Theorem* 1, *if* $g(0) = 0$, *then*

$$\sum_1^{\nu_p}\alpha_\nu^* e^{\lambda_\nu t}-\frac{1}{\pi}\int_0^1 f(u)\frac{\sin r_p(t-u)}{t-u}\,du$$

converges to zero uniformly in (α, β).

For $\alpha_0 = 0$ and

$$\alpha_\nu-\alpha_\nu^* = \frac{g(0)}{\lambda_\nu A'(\lambda_\nu)} = 0.$$

4. Proof of Theorem 2

We have $g(t) = 0$ for all t. In particular, $g(0) = 0$, so that $\alpha_\nu = \alpha_\nu^*$. It suffices to prove that, if γ is any assigned number, then

$$\sum_1^{\nu_p}\alpha_\nu^* e^{\lambda_\nu t}-\frac{1}{\pi}\int_\gamma^{\gamma+1} f(u)\frac{\sin r_p(t-u)}{t-u}\,du$$

converges to zero uniformly in any closed interval $(\gamma+\alpha, \gamma+\beta)$ interior to the open interval $(\gamma, \gamma+1)$. Write $u = v+\gamma$, $f(u) = F(v)$, and

$$\beta_\nu^* = \frac{1}{A'(\lambda_\nu)}\int_0^1 F(v)e^{-\lambda_\nu v}\,dv\int_v^1 k(u)e^{\lambda_\nu u}\,du.$$

Since
$$\int_0^1 k(w)F(w)\,dw = g(\gamma) = 0,$$

we can apply the corollary to the function $F(t)$. It follows that

$$\sum_1^{\nu_p} \beta_\nu^* e^{\lambda_\nu t} - \frac{1}{\pi} \int_0^1 F(v) \frac{\sin r_p(t-v)}{t-v}\, dv$$

converges to zero uniformly in (α, β). Hence

$$\sum_1^{\nu_p} \beta_\nu^* e^{-\lambda_\nu \gamma} e^{\lambda_\nu t} - \frac{1}{\pi} \int_\gamma^{\gamma+1} f(u) \frac{\sin r_p(t-u)}{t-u}\, du$$

converges to zero uniformly in $(\alpha+\gamma, \beta+\gamma)$. It remains to prove that $\beta_\nu^* e^{-\lambda_\nu \gamma} = \alpha_\nu^*$.

For a fixed ν, consider the absolutely continuous function of w,

$$\Phi(w) = \int_0^1 k(u) e^{\lambda_\nu u}\, du \int_w^{w+u} f(s) e^{-\lambda_\nu s}\, ds.$$

Then
$$\Phi(\gamma) = \int_0^1 k(u) e^{\lambda_\nu u}\, du \int_0^u F(v) e^{-\lambda_\nu (v+\gamma)}\, dv$$
$$= A'(\lambda_\nu) \beta_\nu^* e^{-\lambda_\nu \gamma},$$
and
$$\Phi(0) = A'(\lambda_\nu) \alpha_\nu^*.$$
On writing

$$\frac{1}{h}[\Phi(w+h) - \Phi(w)] = \int_0^1 k(u) e^{-\lambda_\nu w} \frac{1}{h} \int_0^h f(w+u+s) e^{-\lambda_\nu s}\, ds du \; - $$
$$- \int_0^1 k(u) e^{\lambda_\nu u}\, du \frac{1}{h} \int_w^{w+h} f(s) e^{-\lambda_\nu s}\, ds,$$

we see that, for almost all w,

$$\Phi'(w) = e^{-\lambda_\nu w}[g(w) - f(w) A(\lambda_\nu)] = 0.$$

Thus $\Phi(\gamma) = \Phi(0)$. This proves Theorem 2.

5. Suppose that the set of distances between pairs of zeros of $A(z)$ has a positive lower bound d. Let $\delta < \frac{1}{2}d$. Let each zero of $A(z)$ be the centre of a disk of radius δ. We denote the disk whose centre is λ_ν by Δ_ν, and its perimeter by γ_ν. We define the contour C'_n to be the circle Γ_n: $|z| = |\lambda_n|$, modified as follows. If Γ_n meets Δ_ν, where $\nu \leqslant n$, we replace the arc of Γ_n which is inside γ_ν by the arc of γ_ν which is outside Γ_n. If Γ_n meets Δ_ν where $\nu > n$, we replace the arc of Γ_n which is inside γ_ν by the arc of γ_ν which is inside Γ_n. Every point of C'_n is at a distance not less than δ from all the zeros of $B(z)$. Hence, as $n \to \infty$, $e^z/B(z)$ is

$O(1)$ on the right half of C_n'; and $1/B(z)$ is $O(1)$ on the left half. Further, C_n' contains only the zeros $\lambda_1,\ldots,\lambda_n$. We apply the argument of § 3, taking C_n' as the path of integration. Let C_n' meet the imaginary axis in $-ir_n'$, ir_n''. Then r_n', r_n'' differ from $r_n = |\lambda_n|$ by not more than δ. Instead of L_1 we now have

$$L_1' = \int\limits_0^1 f(v)\,dv \int\limits_{-ir_n'}^{ir_n''} e^{z(t-v)}\,dz.$$

The deviation of C_n' from circularity will not affect the estimates of § 3. It remains to show that $-ir_n'$, ir_n'' can be replaced by $-ir_n$, ir_n respectively, with an error which is uniformly $o(1)$. Consider, for example, the change from ir_n'' to ir_n. The corresponding change in L_1' is

$$l = \int\limits_0^1 f(v)\,dv \int\limits_{ir_n''}^{ir_n} e^{z(t-v)}\,dz = 2i\int\limits_0^1 f(v)\frac{\sin\sigma_n(t-v)}{t-v}e^{i\rho_n(t-v)}\,dv,$$

where $\sigma_n = \frac{1}{2}(r_n - r_n'')$, $\rho_n = \frac{1}{2}(r_n + r_n'')$.

We may suppose that $\delta < 1$. Then $|\sigma_n| < \frac{1}{2}$. Since $|t-v| < 1$, we can replace $\sin\sigma_n(t-v)/(t-v)$ by a polynomial

$$\sum_0^N \frac{(-1)^m\sigma_n^{2m}(t-v)^{2m}}{(2m+1)!}$$

with an error term less than ϵ if $N \geqslant N(\epsilon)$. Hence, by the Riemann–Lebesgue lemma, l is uniformly $o(1)$ as $n\to\infty$. This proves that in Theorem 1 we can put $\nu_p = p$, and replace r_p by $|\lambda_p|$. In view of the argument of § 4, we can do the same in Theorem 2.

REFERENCE

1. J. Delsarte, 'Les fonctions "moyenne-périodiques"', *J. de Math.* 14 (1938) 403–53.